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A NEW APPROACH FOR THE ANALYTICITY OF THE PRESSURE IN CERTAIN CLASSICAL UNBOUNDED MODELS

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ABSTRACT

We provide a new proof of the analyticity of the free energy that is based on direct C^n -bounds and is suitable for a wide class of classical

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Physics is a human attempt for understanding a certain class of natural phenomena using scientic methods. Mathematical physics is one of these methods or approaches in which a mathematical structure is associated with the structure of a physical phenomenon. The mathematical structure is then studied by mathematical techniques to yield new statements with physical relevance. Applying the methods of mathematics to the investigation of the physical world may be very rewarding. Statistical physics is one these mathematical methods. It attempts to explain the macroscopic behavior of matter on the basis of its microscopic structure. It provides a framework for relating the microscopic properties of individual atoms and molecules to the macroscopic or bulk properties of materials that can be observed in everyday life, therefore explaining thermodynamics as a natural result of statistics and mechanics (classical and quantum) at the microscopic level. The theory of phase transitions is one of the branches of statistical physics in which smoothness and continuity play an important role. In fact phase transitions are characterized mathematically by the degree of non-analyticity of the thermodynamic potentials associated with the given system. There are a number of different thermodynamic potentials that can be used to study mathematically the behavior of a statistical mechanical system. The energy which is stored and is retrievable in the form of work is called the pressure or free energy and is indeed the thermodynamic potential that is commonly used to investigate phase transitions. For many systems in statistical mechanics, the mathematical techniques that are available for proving the analyticity of the free energy are based on complicated indirect arguments and are not suitable for many important models especially the ones with Coulomb interactions. It is therefore imperative to provide a new technique for the analyticity of the free energy that is based on a direct C^n-bound of its nth derivatives. This is precisely the issue that will be addressed in this paper. We propose to provide a proof of the analyticity of the free energy that is based on direct Cn-bounds and is suitable for a wide class of classical unbounded models.

We shall consider systems where each component is located at a site i of a crystal lattice $\Lambda\subset Z^d$, and is described by a continuous real parameter $x_i\in R$. A particular configuration of the total system will be characterized by an element $x=(x_i)i\in\Lambda$ of the product space $\Omega=R^\Lambda$. This set is called the configuration space or phase space.

We shall denote by $\Phi=\Phi^\Lambda$ the Hamiltonian which assigns to each configuration $x\in R^\Lambda$ potential energy (x): The probability measure that describes the equilibrium of the system is then given by the Gibbs measure

$$d\mu^\Lambda(x)=Z_\Lambda^{-1}e^{-\Phi}(x)dx$$

$Z\Lambda>0$ is a normalization constant.

PRELIMINARIES

Let f be a smooth solution of the equation

$$-\Delta f+\nabla\Phi\cdot\nabla f=g-\langle g\rangle_{L^2(\mu^\Lambda)}\ \text{in}\ \mathbb{R}^\Lambda$$

Where Φ is as above and g is a smooth function on $R\Gamma$:
We shall assume that g satisfies

$$|\partial^\alpha\nabla g|\le C_\alpha,\qquad\forall\alpha\in\mathbb{N}^{|\Lambda|}. \qquad (1)$$

Talking gradient on both sides of this equation, we have

$$(-\Delta+\nabla\Phi\cdot\nabla)\nabla f+\mathbf{Hess}\Phi\nabla f=\nabla g\ \text{in}\ \mathbb{R}^\Lambda.$$

The operators

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$$A_\Phi^{(0)} := -\Delta + \nabla\Phi\cdot\nabla$$

and

$$A_\Phi^{(1)} := -\Delta + \nabla\Phi\cdot\nabla + \mathrm{Hess}\Phi$$

are called the Helffer-Sjöstrand operators. They are respectively equivalent to

$$\mathbf{W}_\Phi^{(0)} = \left(-\Delta + \frac{|\nabla\Phi|^2}{4} - \frac{\Delta\Phi}{2}\right)$$

$$\mathbf{W}_\Phi^{(1)} = -\Delta + \frac{|\nabla\Phi|^2}{4} - \frac{\Delta\Phi}{2} + \mathrm{Hess}\Phi.$$

These operators are in some sense deformations of the standard Laplace Beltrami operator. Indeed

$$W_\Phi^{(\cdot)} = e^{-\Phi/2}\circ A_\Phi^{(\cdot)}\circ e^{\Phi/2}$$

and the map (2)

$$U_\Phi:\quad L^2(\mathbb{R}^\Lambda)\to L^2(\mathbb{R}^\Lambda, e^{-\Phi}dx)$$
$$u\longmapsto e^{\frac{\Phi}{2}}u.$$

We shall consider the following hypothesis:

(H): $A_\Phi^{(1)}$ is strictly positive on $L^2(\mathbb{R}^\Lambda, e^{-\Phi}dx)$ i.e. that exixt $c_1>0$ such that

$$A_\Phi^{(1)}\geq c_1.$$

and that the Helfferer-Sjöstrand identity

$$\mathrm{cov}(g,h) = Z^{-1}\int\left(A_\Phi^{(1)^{-1}}\nabla g\cdot\nabla h\right)e^{-\Phi(x)}dx \tag{3}$$

holds.

Definition 1 The lattice support, Sg of a function g on R^Λ is defined here to be the smallest subset Γ of Λ Zd for which g can be written as function of x_l alone with l∈Γ For instance, if g = x_i; Sg = {i}

Now consider the Hamiltonian given by

$$\Phi^t(x) = \Phi_\Lambda(x) - tg(x),$$

where t is a thermodynamic parameter, and g satisfies (1) with Sg ⊊ ΛThe free energy of the system is defined by

$$P_\Lambda(t) = \frac{1}{|\Lambda|}\log\left[\int_{\mathbb{R}^\Lambda} dx e^{-\Phi^t(x)}\right].$$

We have proved in Lo. A., (2007) the following formula: There exists T*> 0 such that for all t∈ [0, T*);

$$F_\Lambda^{(n)}(t) = (n-1)!\left\langle A_g^{n-1}g\right\rangle_{t,\Lambda}\quad n\geq 1,$$

where

$$F_\Lambda(t) = |\Lambda|\,P_\Lambda(t),$$

$$A_g h := A_{\Phi_\Lambda^t}^{(1)^{-1}}\nabla h\cdot\nabla g,$$

$$\langle\cdot\rangle_{t,\Lambda} = \frac{\int_{\mathbb{R}^\Lambda}\cdot dx e^{-\Phi^t(x)}}{\int_{\mathbb{R}^\Lambda} dx e^{-\Phi^t(x)}}$$

C^n-BOUNDS FOR THE ANALYTICITY OF THE FREE ENERGY

It is well known that if p(t) is an ininitely differentiable function de.ned on an open set D⊂R; then f is real analytic if and only if for every compact set K⊂D; there exists a constant C such that for every t∈K

$$\left|p^{(n)}(t)\right|\leq C^{n+1}n!$$

We now propose to establish a bound of this type fo $p(t) := \lim_{\Lambda\text{-}Z^d} P\Lambda(t)$

Using our formula (4), we first prove the following Lemma:

Lemma 1 Assume that satisfies (***H***), and g satisfies (1) with Sg ⊊ Λ (1) Then

$$\sum_{k=0}^{n-1}\frac{\langle g^k\rangle_{\Lambda,t}\langle A_g^{n-k-1}g\rangle_{\Lambda,t}}{k!} = \frac{1}{(n-1)!}\langle g^n\rangle_{\Lambda,t},\qquad n\geq 1.$$

Proof first Observe that

$$\begin{aligned}
&\langle g^p A_g h\rangle_{\Lambda,t}\\
&= \left\langle g^p A_{\Phi^t}^{(1)^{-1}}\nabla h\cdot\nabla g\right\rangle_{\Lambda,t}\\
&= \frac{1}{p+1}\left\langle A_{\Phi^t}^{(1)^{-1}}\nabla h\cdot\nabla g^{p+1}\right\rangle_{\Lambda,t}\\
&= \frac{1}{p+1}\mathrm{cov}\left(g^{p+1},h\right)\\
&= \frac{1}{p+1}\left[\langle g^{p+1}h\rangle_{\Lambda,t} - \langle g^{p+1}\rangle_{\Lambda,t}\langle h\rangle_{\Lambda,t}\right],\quad p=0,1,\ldots
\end{aligned}$$

Setting

$$k = p+1\quad\text{and}\quad h = A_g^{n-k-1}g,$$

yields

$$\langle g^k\rangle_{\Lambda,t}\langle A_g^{n-k-1}g\rangle_{\Lambda,t} = \langle g^k A_g^{n-k-1}g\rangle_{\Lambda,t} - k\langle g^{k-1}A_g^{n-k}g\rangle_{\Lambda,t}.$$

Now dividing by k!, summing over k and noticing that on the right hand side one obtains a telescoping sum, yields

$$\sum_{k=0}^{n-1}\frac{\langle g^k\rangle_{\Lambda,t}\langle A_g^{n-k-1}g\rangle_{\Lambda,t}}{k!} = \frac{1}{(n-1)!}\langle g^n\rangle_{\Lambda,t}.$$

We now propose to deconvolve the convolution equation in the Lemma above via the discrete Laplace transform commonly called the *z*- transform. Recall that given the sequence $\{x_n\}_{n=0}^{\infty}$, the *z*-transform of $\{x_n\}_{n=0}^{\infty}$, is defined as follows

$$Z\left(\{x_n\}_{n=0}^{\infty}\right) = \sum_{n=0}^{\infty} x_n z^{-n} := X_{\Lambda}(z)$$

which is a series involving powers of 1/*z*. The z-transform region of convergence for the Laurent series is chosen to be

$$|z| > R, \qquad \text{where } R = \limsup \sqrt[n]{|x_n|}.$$

The inverse *z*-transform is given by

$$x_n = Z^{-1}[X(z)] = \frac{1}{2\pi i}\oint_C X(z) z^{n-1} dz,$$

where *C* is any positively oriented simple closed curve that lies in the region of convergence and winds around the origin.We also have the following convolution property: if $\{x_n\}_{n=0}^{\infty}$, and $\{y_n\}_{n=0}^{\infty}$ are two sequences with z-transform $X_{\Lambda}(z)$and $Y_{\Lambda}(z)$ are two sequences with z-transform X(z) and Y(z) respectively, then

$$Z(x_n * y_n) = X_{\Lambda}(z) Y_{\Lambda}(z)$$

where $x_n * y_n$ is defined as the convolution sum

$$x_n * y_n = \sum_{k=0}^{n} x_k y_{n-k}.$$

now let

$$x_n = \frac{\langle g^n \rangle_{t,\Lambda}}{n!} \quad \text{and} \quad y_n = \langle A_g^n g \rangle_{t,\Lambda},$$

and assume that

$$R = \limsup_n \left| \frac{\langle g^n \rangle_{t,\Lambda}}{n!} \right|^{1/n} < \infty.$$

(6) is equivalent to

$$x_n * y_n = (n+1) x_{n+1}.$$

Denote by $X_{\Lambda}(z)$ and $Y_{\Lambda}(z)$ the *z*-transform of the sequences $\{x_n\}_{n=0}^{\infty}$, and $\{y_n\}_{n=0}^{\infty}$ respectively. Applying the z-transform to this convolution equation above and using the properties of z-transform, we obtain

$$X_{\Lambda}(z) Y_{\Lambda}(z) = -z^2 \frac{dX_{\Lambda}(z)}{dz}.$$

Thus

$$Y_{\Lambda}(z) = -z^2 \frac{X'_{\Lambda}(z)}{X_{\Lambda}(z)}.$$

Now using the inverse *z*-transform, we get

$$\langle A_g^n g \rangle_{t,\Lambda} = -\frac{1}{2\pi i} \oint_C \frac{X'_{\Lambda}(z)}{X_{\Lambda}(z)} z^{n+1} dz.$$

we have almost prooved:

Proposition 2 Suppose the Hamiltonian Φ satisfies the assumption (***H***) above and g satisfies (1)

$$R = \limsup_n \left| \frac{\langle g^n \rangle_{t,\Lambda}}{n!} \right|^{1/n} < \infty$$

and that the thermodynamic limit

$$G(z) = \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \frac{X'_{\Lambda}(z)}{X_{\Lambda}(z)} \ \textit{exists}$$

where

$$X_{\Lambda}(z) = \sum_{n=0}^{\infty} \frac{\langle g^n \rangle_{t,\Lambda}}{n!} z^{-n}$$

with $S_g \subsetneq \Lambda$, Then $\left| p^{(n)}(t) \right| \leq \lambda (n-1)! L^n,$

where *λ* and *L* are positive constants that are independent of Λ. Thus the pressure is analytic in the thermodynamic limit.

Proof. Form the equations

$$\langle A_g^n g \rangle_{t,\Lambda} = -\frac{1}{2\pi i} \oint_C \frac{X'_{\Lambda}(z)}{X_{\Lambda}(z)} z^{n+1} dz.$$

and

$$F_{\Lambda}^{(n)}(t) = (n-1)! \langle A_g^{n-1} g \rangle_{t,\Lambda},$$

we have

$$F_{\Lambda}^{(n)}(t) = -\frac{1}{2\pi i}(n-1)! \oint_C \frac{1}{|\Lambda|} \frac{X'_{\Lambda}(z)}{X_{\Lambda}(z)} z^n dz.$$

Now taking the thermodynamic limit, we have

$$\left| p^{(n)}(t) \right| \leq \frac{1}{2\pi}(n-1)! \oint_C |G(z) z^n| \, |dz|.$$

If L is the radius of the smallest circle *C* containing the poles of G(*z*); then we get

APPLICATIONS

A) The result of Proposition 2 is suitable for unbounded Hamiltonians discussed by Bach et al., (2000); Bach and Moller, (2003) and Bach, V., Moller, J.S., (2004). The exponential decay of the two point correlation function is also a consequence of Proposition 2 (see remark 2 below).

B) The result above may be applied to non-convex Hamiltonians satisfying:

1. $\lim_{|x|\to\infty} |\nabla\Phi(x)| = \infty$
2. $\exists M$, any $\partial^\alpha \Phi$ with $|\alpha| = M$ is bounded on $\mathbb{R}^\Lambda$.
3. $\forall\, |\alpha| \geq 1$, $|\partial^\alpha \Phi(x)| \leq C_\alpha \left(1 + |\nabla\Phi(x)|^2\right)^{1/2}$ for some $C_\alpha > 0$
4. $\exists w > 0, C > 0$ such that $x \cdot \nabla\Phi \geq C|x|^{1+w}$ for all $|x| \geq \frac{1}{C}$.

as discussed in Jhonsen J., (2000).. Here $|x| = (\sum_{i\in\Lambda} x_i^2)^{1/2}$ and in what follows $\alpha = (\alpha_i)_{i\in\Lambda} \in \mathbb{Z}_+^{|\Lambda|}$ shall donate a multiindex. We set

$$|\alpha| = \sum_{i\in\Lambda} \alpha_i, \alpha! = \alpha_1! \cdots \alpha_n!.$$

If $\beta = (\beta_i)_{i\in\Lambda} \in \mathbb{Z}_+^{|\Lambda|}$ and $\beta_j \leq \alpha_j$ for all j 2 ; then we write For all j , then we write . For $\beta \in \mathbb{Z}_+^{|\Lambda|}$ such that we put

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}.$$

If $\alpha = (\alpha_i)_{i\in\Lambda} \in \mathbb{Z}_+^{|\Lambda|}$ we write $x \in \mathbb{R}^\Lambda$ $x^\alpha = \prod_{i\in\Lambda} x^{\alpha_i}$, and

$$\partial^\alpha = \partial^{\alpha_1}/\partial x_1^{\alpha_1} ... \partial^{\alpha_m}/\partial x_m^{\alpha_m},$$

where $m = |\Lambda|$. For instance

$$\Phi(x) = \frac{1}{h}\sum_{j=1}^{n}\left(\frac{\lambda}{12}x_j^4 + \frac{\nu}{2}x_j^2\right) + \frac{1}{h}\frac{\mathcal{J}}{2}\sum_{j=1}^{n}|x_j - x_{j+1}|^2, \quad (7)$$

where $x_{n+1} = x_1$, $h > 0$, $\mathcal{J} > 0$, and $\lambda > 0 > \nu$. These models are commonly used in Euclidean .eld theory. They have unbounded second derivatives and satisfy assumption 1-4. Indeed, first observe that there exists $j \in \{1, ..., n\}$ such that $x_j \geq \frac{|x|}{\sqrt{n}}$ otherwise one would have $|x|^2 < |x|^2$ using this, it is clear that

$$\begin{aligned} x \cdot \nabla\Phi &\geq \frac{1}{h}\sum_{j=1}^{n}\left(\frac{\lambda}{3}x_j^4 + \nu x_j^2\right) \\ &= \frac{|x|^4}{h}\left[\frac{\lambda}{3}\left(\frac{1}{\sqrt{n}}\right)^4 - |\nu||x|^{-2}\right] \\ &\geq \frac{|x|^4}{C} \text{ when } |x| \geq C \text{ for some sufficiently large } C. \end{aligned}$$

C) Another example of models satisfying assumptions 1-4 is given by the Kaac model

$$\Phi(x) = \frac{x^2}{2} - 2\sum_{i\sim j} \ln\cosh\left[\sqrt{\frac{\beta}{2}}(x_i + x_j)\right].$$

which is a mean field model introduced by Kac, (1966) in an effort to study rigorously certain problems of phase transition and in particular to justify the van der Waals theory of liquid-vapor transition. The exact model is analogous to the two dimensional Ising model and constructed as follows:

Let J be an even positive lipschitz function satisfying

$$\int_{\mathbb{R}} J(r)dr = 2.$$

define the family $\{J\}_{\gamma>0}$ by

$$\forall r \in \mathbb{R}, \quad J_\gamma(r) = \gamma J(\gamma r).$$

The choice made in Kac, (1966) consisted of

$$J(r) = e^{-|r|}.$$

For a fixed $\gamma>0$, one defines an interaction potential J on $Z^2 \times Z^2$ by

$$\mathbb{J}_\gamma(k,l,\tilde{k},\tilde{l}) = J_\gamma(k-\tilde{k})\mathcal{J}(l,\tilde{l})$$

with

$$\mathcal{J}(l,\tilde{l}) = \delta_{l,\tilde{l}} + \frac{1}{2}\left(\delta_{l,\tilde{l}+1} + \delta_{l,\tilde{l}-1}\right).$$

Here $\delta_{i,j}$ is the Kronecker delta function.

Let Λ be a finite subset of Z^2; the Hamiltonian of the configuration $\sigma_\Lambda = (\sigma_i)_{i\in\Lambda} \in \{-1,1\}^\Lambda$ with boundary condition $\sigma_{\Lambda^c} = (\sigma_i)_{i\in\Lambda^c}$ is given by

$$H_{\Lambda,\gamma}(\sigma_\Lambda/\sigma_{\Lambda^c}) = -\frac{1}{2}\sum_{i,j\in\Lambda}\mathbb{J}_\gamma(i,j)\sigma_i\sigma_j - \sum_{i\in\Lambda, j\in\Lambda^c}\mathbb{J}_\gamma(i,j)\sigma_i\sigma_j.$$

Kac showed in Kac, (1966) that when

$$J(r) = e^{-|r|},$$

this model may be studied through the transfer operator

$$K_\gamma^m = e^{-\frac{1}{2}\gamma q(x)} e^{\gamma\Delta_m} e^{-\frac{1}{2}\gamma q(x)}$$

where

$$\gamma q(x) = \frac{1}{2}\tanh\left(\frac{\gamma}{2}\right)\sum_{i=1}^{m}x_i^2 - \sum_{i=1}^{m}\log\cosh\left[\sqrt{\frac{\gamma\beta}{2}}(x_i + x_{i+1})\right]$$

with the convention $x_{m+1}=x_1$: He proved that when γ approaches 0; the behavior of the system only depends on the Kac potential

$$q(x) = \sum_{i=1}^{m}\frac{x_i^2}{4} - \sum_{i=1}^{m}\log\cosh\left[\sqrt{\frac{\beta}{2}}(x_i + x_{i+1})\right]$$

Thus by reducing the two dimensional problem into a one dimensional problem, M. Kac showed that the critical temperature occurs at $\beta_c = \frac{1}{4}$.

The mean field Kac Hamiltonian

$$\Phi(x) = \frac{x^2}{2} - 2\sum_{i\sim j}\ln\cosh\left[\sqrt{\frac{\beta}{2}}(x_i + x_j)\right]$$

satis.es assumptions 1-4 above if $\beta < \frac{1}{4d}$.

Indeed let $\Phi(x) = \frac{x^2}{2} + \Psi(x)$, where

$$\Psi(x) = -2\sum_{i\sim j} \ln\cosh\left[\sqrt{\frac{\beta}{2}}(x_i + x_j)\right].$$

we have

$$\Psi_{x_i} = -2\sum_{j:j\sim i} \frac{\sqrt{\frac{\beta}{2}}\sinh\left[\sqrt{\frac{\beta}{2}}(x_i + x_j)\right]}{\cosh\left[\sqrt{\frac{\beta}{2}}(x_i + x_j)\right]}$$

$$\Psi_{x_i x_k} = \begin{cases} -\beta\sum_{j:j\sim i}\frac{1}{\cosh^2\left[\sqrt{\frac{\beta}{2}}(x_i + x_j)\right]} & \text{if } k = i \\ -\frac{\beta}{\cosh^2\left[\sqrt{\frac{\beta}{2}}(x_i + x_k)\right]} & \text{if } k \sim i \\ 0 & \text{otherwise} \end{cases}$$

It then follows that

$$|\Psi_{x_i}| \le 4d\sqrt{\frac{\beta}{2}},$$

and

$$|\Psi_{x_i x_i}| \le 2d\beta,$$

$$|\Psi_{x_i x_k}| \le \beta \qquad \text{if } k \sim i.$$

Similarly, using the properties of cosh and sinh and the fact that sinh t ≤ cost t for all t one can see that all derivatives of order greater than or equal to one are bounded. Now we propose to .nd the values of for which assumption 4 holds. i.e. there exist $w > 0, C > 0$ such that $x \cdot \nabla\Phi \ge C|x|^{1+w}$ for all $|x| \ge \frac{1}{2}$.

First write

$$\begin{aligned}\Psi_{x_i} &= \int_0^1 \frac{d}{ds}\Psi_{x_i}(sx)ds = \int_0^1 x\cdot\nabla\Psi_{x_i}(sx)ds \\ &= \sum_{j\in\Lambda}\int_0^1 \Psi_{x_i x_j}(sx)x_j ds.\end{aligned}$$

Thus

$$x\cdot\nabla\Psi = \sum_{i,j\in\Lambda} c_{ij}x_i x_j,$$

where

$$c_{ij} = \int_0^1 \Psi_{x_i x_j}(sx)ds.$$

There is a Schur's Lemma (Steel, (2004)) that says for each rectangular array $(c_{ij})_{1\le i\le m,\ 1\le j\le n}$ and each pair of sequence $(x_i)_{1\le i\le m}$ and $(y_j)_{1\le j\le n}$ we have the bound

$$\left|\sum_{i=1}^m\sum_{j=1}^n c_{ij}x_i y_j\right| \le \sqrt{RC}\left(\sum_{i=1}^m |x_i|^2\right)^{1/2}\left(\sum_{j=1}^n |y_j|^2\right)^{1/2}$$

where *R* and *C* are the row sum and column sum maxima defined by

$$R = \max_i \sum_{j=1}^n |c_{ij}| \quad \text{and} \quad C = \max_j \sum_{i=1}^m |c_{ij}|.$$

Now using this result, we have

$$\begin{aligned}\sum_{j\in\Lambda}|c_{ij}| &\le \sum_{j\in\Lambda}\int_0^1 |\Psi_{x_i x_j}|(sx)ds \\ &= \int_0^1 |\Psi_{x_i x_i}|(sx)ds + \sum_{j\in\Lambda, j\sim i}\int_0^1 |\Psi_{x_i x_j}|(sx)ds \\ &\le 2d\beta + 2d\beta.\end{aligned}$$

Thus $R \le 4d\beta$.

Similarly, we have $C \le 4d\beta$.

and $|x\cdot\nabla\Psi| \le 4d\beta|x|^2$.

It then follows that $x\cdot\nabla\Phi \ge |x|^2(1 - 4d\beta)$.

Thus if $\beta < \frac{1}{4d}$.

Remark 1 The problem of .nding a direct proof of convergence of the Mayer expansion for dipoles at low activity (which does not use cluster expansions) has been open for a very long time see Procacci, et al., (1997). We believe that if the thermodynamic parameter t plays the role of the activity, then the technique developed in this paper may be suitable for solving this problem.

Indeed, because the dipole-dipole interaction should be smoothed out at short distance so that it is stable, one only needs to find a suitable regularization of the dipole-dipole potential that satisfies assumption (H).

The following lemma is needed to support the argument that will be given in Remark 2. A more restricted version can be found in Helffer and Sjöstrand, (1994).

Lemma 3 (FKG inequality) If is such that the associated Witten Laplacian on one forms $A_\Phi^{(1)}$ is strictly positive and g, h are two monotone increasing functions on R^Λ satisfying (1); then cov(g,h)≥0.

Proof We have

$$\mathbf{cov}(g,h) = Z^{-1}\int\left(A_\Phi^{(1)^{-1}}\nabla g\cdot\nabla h\right)e^{-\Phi(x)}dx.$$

Let $A_{\Phi}^{(1)^{-1}} \nabla g = w = (w_j)_{j \in \Lambda}$.

we only need to prove that consider equation $g_{x_j} > 0 \Longrightarrow w_j > 0.$

$$A_{\Phi}^{(1)} w = \nabla g.$$

Write $w = w^+ - w^-$ where $w_j^+ = \sup\{0, w_j\}$ and $w_j^- = \sup\{0, -w_j\}$

We have $\nabla g = A_{\Phi}^{(1)} w^+ - A_{\Phi}^{(1)} w^-$

multiplying both sides by w^- and integrating with respect to $e^{-\Phi(x)} dx$ we get

$$\left\langle \nabla g, w^- \right\rangle = \left\langle A_{\Phi}^{(1)} w^+, w^- \right\rangle - \left\langle A_{\Phi}^{(1)} w^-, w^- \right\rangle$$

The strict positivity of $A_{\Phi}^{(1)}$ implies that the right hand side of this last above inequality is negative. Thus we have

$$0 \le \left\langle \nabla g, w^- \right\rangle \le 0 \qquad \Longrightarrow \left\langle \nabla g, w^- \right\rangle = 0$$

Hence, w^-=0 and the result follows.

Remark 2. Proposition 2 could also be used to provide a simple proof of the exponential decay of the two-point correlations that does not use estimate for the spectral gap of the Witten Laplacian on one forms in the one dimensional case. Indeed, assume that g is explicitly given by

$$g(x) = \sum_{i \in \Gamma} \xi_i x_i \quad \text{where } \xi_i = e^{\max_{j \in \Gamma,\, j \neq i} d(i,j)} \text{ and } \Gamma \subsetneq \Lambda.$$

The assumptions on g are satis.ed in the one dimensional case d = 1: Helffer and Sjöstrand,(1994).

We have

$$P_{\Lambda}(t) = \frac{1}{|\Lambda|} \log \left[\int_{\mathbb{R}^{\Lambda}} dx e^{-\Phi(x) + t \sum_{i \in \Gamma} \xi_i x_i} \right]$$

$$P'_{\Lambda}(t) = \frac{1}{|\Lambda|} \sum_{i_1 \in \Gamma} \xi_{i_1} \langle x_{i_1} \rangle_{t,\Lambda}.$$

$$\begin{aligned} P''_{\Lambda}(t) &= \frac{1}{|\Lambda|} \sum_{i_1 \in \Gamma} \xi_{i_1} \mathbf{cov} \left(x_{i_1}, \sum_{i_2 \in \Gamma} \xi_{i_2} x_{i_2} \right) \\ &= \frac{1}{|\Lambda|} \sum_{i_1, i_2 \in \Gamma} \xi_{i_1} \xi_{i_2} \mathbf{cov} \left(x_{i_1}, x_{i_2} \right) \end{aligned}$$

Using Proposition 2 with n = 2, we get

$$\left| \sum_{i_1, i_2 \in \Gamma} \xi_{i_1} \xi_{i_2} \mathbf{cov} \left(x_{i_1}, x_{i_2} \right) \right| \le C,$$

where *C* is a constant that is independent of Λ Now observe that

$$\xi_{i_k} = e^{\max_{j \in \Gamma,\, j \neq i_k} d(i_k, j)} \ge e^{\max_{j_p:\, p = 1, \dots, n,\; j_p \neq i_k} d(i_k, j_p)}$$

for all k = 1 .....n

and that cov$(x_{i_1}, x_{i_2}) \ge 0$ by Lemma 4. Thus we have cov $(x_{i_1}, x_{i_2}) \le C e^{-2d(i_1, i_2)}$.

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REFERENCES

Bach V., Jecko T. and Sjöstrand, J., 2000. Correlation asymptotics of classical lattice spin systems wnonconvex Hamiltonian function at low temperature. Ann. Henri Poincare, **1**: 59-100.

Bach V. and Moller, J.S., 2003. Correlation at low temperature I, exponential decay.J. Funct. Anal., **203**: 93-148.

Bach V. and Moller J.S., 2004. Correlation at low temperature II, Asymptotics J. of Stat. Phys., **116**: 114.

Helffer B. and Sjöstrand, J., 1994. On the correlation for Kac-like models in the convex case.J. of Stat. Phys, **74** Nos.1/2.

Johnsen J., 2000. On spectral properties of Witten-Laplacians, their range of projections and and Brascamp-Lieb inequality Integral Equations Operator Theory:36.

Kac M., 1966. Mathematical mechanism of phase transitions. Gordon & Breach, New York, **3**: 288-324.

Lo. A., 2007. On the exponential decay of the n-point correlation functions and the analyticity of the pressure J. Math. Phys. **48**, 123506.

Procacci A., Pereira E. and Al, Armando, G. M. Neves and Domingos H. U. Marchetti, 1997. Coulomb interaction symmetries and the Mayer series in the two-dimensional dipole gas ournal of Statistical Physics Volume 87, Numbers 3-4, 877-889, DOI: 10.1007/BF02181248.

Steel M., 2004. The Cauchy-Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities Maa Problem Books Series ISBN-10: 052154677X ISBN-**13**: 978-0521546775.