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UNEQUAL PARTITE SETS IN BIPARTITE TREES WITH USE OF LINEAR ALGEBRA IN CRYPTOGRAPHY

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ABSTRACT

In this paper, labelled Bipartite Trees with both unequal and equal partite sets and employing certain graph labelling schemes namely harmonious, graceful, sequential and felicitous are being presented for encryption and decryption. Some modified modulo operations have been incorporated with these methods to develop the Cryptography algorithms for greater security. Certain other mathematical concepts involving matrices and concepts from Linear Algebra also have been applied in developing Cryptography algorithms for data transfer with a significantly greater security, all these methods employing the labelled Bipartite Trees.

KEYWORDS: Label Matrix, Harmonious Labelling, Felicitous Labelling, Graceful Labelling, Sequential Labelling

Application of graph labelling schemes, namely inner magic and inner antimagic in Cryptography have been made in (Krishnaa, 2019). These labellings were discovered in (Krishnaa and Dulawat, 2006). Tokareva, (2014) has seen the relationship between concepts of Graph Theory and Cryptography. Chase and Kamara (2010) has also discussed the encryption of cryptography. Krishnaa (2021) has given the application of some particular labelled graphs in cryptography. Sudarsana et al., (2020) has given the super mean and magic labelling schemes in cryptography. Baskar Babujee and Babitha (2012) has discussed both encryption and decryption regarding labelled graphs. Maheswari et al., (2020) has used star graphs for encryption and secret coding of messages. Krishnaa (2018) has given directions of using labelled graphs in cryptography and other applications. Krishnaa (2004) gives the harmonious, felicitous, sequential and graceful labelling of Bipartite Trees.

Methods with adding of vertex labels and induced edge labels, two modified modulo operations, Matrix operations such as Adjoint, matrix multiplication such as squaring of a matrix and Matrix Transformation, Change of Basis etc. have been used in this work. These methods have been developed for Bipartite Trees with unequal partite sets for the four kinds of labelling schemes namely harmonious, felicitous, sequential and graceful which are as per the algorithms given in Krishnaa (2004). Unequal partite sets can render still greater privacy and security than equal partite sets.

MATERIALS AND METHODS

Methods used are from Graph Theory and other mathematical methods such as concepts from Linear Algebra and used in applications for Cryptography. Methods using adding of vertex labels and induced edge labels, matrix operations and concepts of Linear Algebra have been used in this work.

Fundamental Definitions

Bipartite Tree: A bipartite graph is a graph whose vertices can be divided into two disjoint and independent sets U&V such that every edge connects one vertex in U to one vertex in V. If vertex sets U and V are having equal number of elements they are being termed here having Equal Partite Sets, otherwise having Unequal Partite Sets.

Adjoint of a matrix: Let $A = [a_{ij}]$ be a square matrix of order n. The adjoint of a matrix A is the transpose of the cofactor matrix of A. It is denoted by adj A. An adjoint matrix is also called an adjugate matrix.

Matrix Transformation: Let A be an m×n matrix. The matrix transformation associated with A is the transformation:

$T : R^n \rightarrow R^m$ defined by $T(x) = Ax$.

Label Matrix: Depending on the type of graph labelling used, these matrices have been newly developed such as Harmonious Label Matrix for the harmonious labelling used for the Bipartite Tree. The entries of this type of matrix are the induced edge labels. For instance, a vertex

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is labelled in the Left Partite Set with i and with j in the Right Partite Set, then the matrix entry a_{ij} is the induced edge label of the edge formed by the end vertices labeld with i and j in the left and right respectively. Matrix (a) shows the Harmonious Label Matrix for the Harmonious Tree of Figure 1.

RESULTS AND DISCUSSION

Unequal Partite Sets in a Bipartite Tree (For greater Privacy and Secrecy) ($l \neq r$): Due to the unequal number of vertices in left and right partite sets, a Plain text of 3 letters can get converted to even 5 or 6 or more letters of Cipher text depending on the Bipartite Tree drawn, thus hiding the Plain text still more and providing greater privacy and security as will be shown in the Illustration.

Harmonious Labeled Tree G(7, 6)

We have Plain Text "ACT" which have to be converted into a Cipher Text with the help of Harmonious Labelled Tree G(7, 6) as shown in Figure 1. It has 3 elements in left partite set and 4 elements in right partite set.

Method 1: Adding vertex labels and induced edge labels.

Illustration:

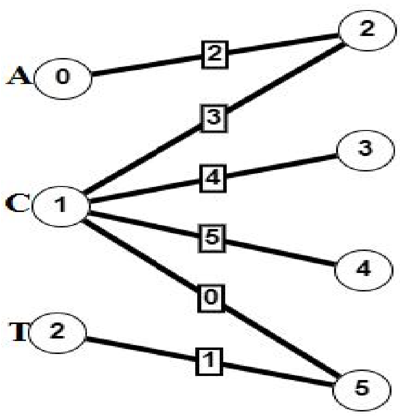

Figure 1: Harmonious Labelled Tree

Scheme 1 (Add vertex label)

A+2 =C, C+2 = E, C+3 = F, C+4 = G, C+5 = H, T+5 = Y.

Cipher Text is: CEFGHY

Scheme 2 (Add induced edge label)

A+2 = C, C+3 = F, C+4 = G, C+5 = H, C+0 =C, T+1 = U.

Cipher Text is: CFGHCU

Felicitous Labeled Tree:

Scheme 1 (Add vertex label)

A+3 =D, C+3 = F, C+4 = G, C+5 = H, C+6 = I, T+6 = Z.

Cipher Text is: DFGHIZ

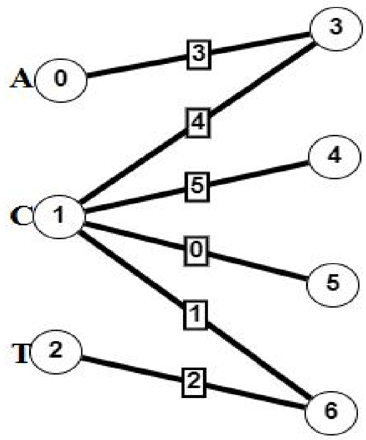

Figure 2: Felicitous Labelled Tree

Scheme 2 (Add induced edge label)

A+3 = D, C+4 = G, C+5 = H, C+0 =C, C+1 = D, T+2 = V

Cipher Text is: DGHCDV

Sequential Labelled Tree:

Scheme 1 (Add vertex label)

A+2 =C, C+2 = E, C+3 = F, C+4 = G, C+5 = H, T+5 = Y.

Cipher Text is: CEFGHY

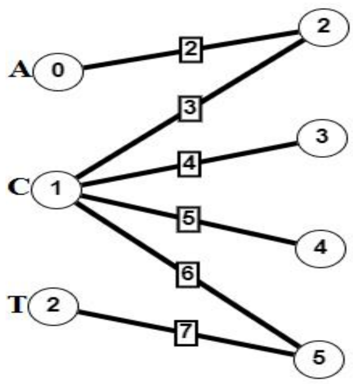

Figure 3: Sequential Labelled Tree

Scheme 2 (Add induced edge label)

A+2 = C, C+3 = F, C+4 = G, C+5 = H, C+6 =I, T+7 = A.

Cipher Text is: CFGHIA.

Graceful Labelled Tree:

Scheme 1 (Add vertex label)

A+6 = G, C+6 = I, C+5 = H, C+4 = G, C+3 =F, T+3 = W.

Cipher Text is: GIHGFW

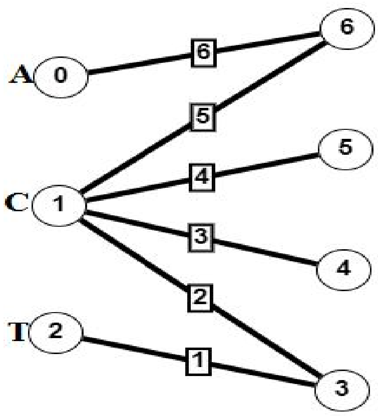

Figure 4: Graceful Labelled Tree

Scheme 2 (Add induced edge label)

A+6 = G, C+5 = H, C+4 = G, C+3 = F, C+2 =E, T+1 = U.

Cipher Text is: GHGFEU.

Method 2: (Modified Modulo Operations)

If p is the number of vertices and q is the number of edges of graph G(p, q) we will apply the following operations on vertex labeling to get new Cipher Text. $\oplus_q$ is addition modulo q where q is the number of edges of a Bipartite Tree and $\odot_q$ is the multiplication modulo q.

i. $(a \oplus_q b) \oplus_q (a \odot_q b)$

ii. $(a \oplus_q b) \odot_q (a \odot_q b)$

Where 'a' is the vertex labeling from the left partite set and 'b' is the vertex labeling from the right partite set connected to vertex 'a' (a and b are the end vertices).

Encryption: Send the type of operation, the Bipartite Tree in the form of adjacency matrix or Label Matrix and the Cipher text to the Receiver.

Decryption: The Receiver does the operation in reverse to get the Plain text.

We take Plain Text as "ACT" and will get Cipher text with the help of modified operations.

Harmonious Labelled Tree

From the figure 1, For G(7, 6), 'a' will be from left partite set i.e. {0, 1, 2} and 'b' will be from right partite set i.e. {2, 3, 4, 5}.

(Operation 1): Plain Text is: ACT

$(0\oplus_6 2)\oplus_6(0\odot_6 2) = 2 \rightarrow$ A+2 = C

$(1\oplus_6 2)\oplus_6(1\odot_6 2) = 5 \rightarrow$ C+5 = H

$(1\oplus_6 3)\oplus_6(1\odot_6 3) = 1 \rightarrow$ C+1 = D

$(1\oplus_6 4)\oplus_6(1\odot_6 4) = 3 \rightarrow$ C+3 = F

$(1\oplus_6 5)\oplus_6(1\odot_6 5) = 5 \rightarrow$ C+5 = H

$(2\oplus_6 5)\oplus_6(2\odot_6 5) = 5 \rightarrow$ T+5 = Y

Cipher Text is: CHDFHY.

(Operation 2):

$(0\oplus_6 2)\odot_6(0\odot_6 2) = 0 \rightarrow$ A+0 = A

$(1\oplus_6 2)\odot_6(1\odot_6 2) = 0 \rightarrow$ C+0 = C

$(1\oplus_6 3)\odot_6(1\odot_6 3) = 0 \rightarrow$ C+0 = C

$(1\oplus_6 4)\odot_6(1\odot_6 4) = 2 \rightarrow$ C+2 = E

$(1\oplus_6 5)\odot_6(1\odot_6 5) = 0 \rightarrow$ C+0 = C

$(2\oplus_6 5)\odot_6(2\odot_6 5) = 4 \rightarrow$ T+4 = X

Cipher Text is: ACCECX.

Felicitous Labelled Tree:

From the figure 2, For G(7, 6), 'a' will be from left partite set i.e. {0, 1, 2} and 'b' will be from right partite set i.e. {3, 4, 5, 6}.

(Operation 1): Plain Text is: ACT

$(0\oplus_6 3)\oplus_6(0\odot_6 3) = 3 \rightarrow$ A+3 = D

$(1\oplus_6 3)\oplus_6(1\odot_6 3) = 1 \rightarrow$ C+1 = D

$(1\oplus_6 4)\oplus_6(1\odot_6 4) = 3 \rightarrow$ C+3 = F

$(1\oplus_6 5)\oplus_6(1\odot_6 5) = 5 \rightarrow$ C+5 = H

$(1\oplus_6 6)\oplus_6(1\odot_6 6) = 1 \rightarrow$ C+1 = D

$(2\oplus_6 6)\oplus_6(2\odot_6 6) = 2 \rightarrow$ T+2 = V

Cipher Text is: DDFHDV.

(Operation 2):

$(0\oplus_6 3)\odot_6(0\odot_6 3) = 0 \rightarrow$ A+0 = A

$(1\oplus_6 3)\odot_6(1\odot_6 3) = 0 \rightarrow$ C+0 = C

$(1\oplus_6 4)\odot_6(1\odot_6 4) = 2 \quad \rightarrow \quad C+2 = E$

$(1\oplus_6 5)\odot_6(1\odot_6 5) = 0 \quad \rightarrow \quad C+0 = C$

$(1\oplus_6 6)\odot_6(1\odot_6 6) = 0 \quad \rightarrow \quad C+0 = C$

$(2\oplus_6 6)\odot_6(2\odot_6 6) = 0 \quad \rightarrow \quad T+0 = T$

Cipher Text is: ACECCT.

Sequential Labelled Tree

From the figure 3, For G(7, 6), 'a' will be from left partite set i.e. {0, 1, 2} and 'b' will be from right partite set i.e. {2, 3, 4, 5}.

(Operation 1): Plain Text is: ACT

$(0\oplus_6 2)\oplus_6(0\odot_6 2) = 2 \quad \rightarrow \quad A+2 = C$

$(1\oplus_6 2)\oplus_6(1\odot_6 2) = 5 \quad \rightarrow \quad C+5 = H$

$(1\oplus_6 3)\oplus_6(1\odot_6 3) = 1 \quad \rightarrow \quad C+1 = D$

$(1\oplus_6 4)\oplus_6(1\odot_6 4) = 3 \quad \rightarrow \quad C+3 = F$

$(1\oplus_6 5)\oplus_6(1\odot_6 5) = 5 \quad \rightarrow \quad C+5 = H$

$(2\oplus_6 5)\oplus_6(2\odot_6 5) = 5 \quad \rightarrow \quad T+5 = Y$

Cipher Text is: CHDFHY.

(Operation 2):

$(0\oplus_6 2)\odot_6(0\odot_6 2) = 0 \quad \rightarrow \quad A+0 = A$

$(1\oplus_6 2)\odot_6(1\odot_6 2) = 0 \quad \rightarrow \quad C+0 = C$

$(1\oplus_6 3)\odot_6(1\odot_6 3) = 0 \quad \rightarrow \quad C+0 = C$

$(1\oplus_6 4)\odot_6(1\odot_6 4) = 2 \quad \rightarrow \quad C+2 = E$

$(1\oplus_6 5)\odot_6(1\odot_6 5) = 0 \quad \rightarrow \quad C+0 = C$

$(2\oplus_6 5)\odot_6(2\odot_6 5) = 4 \quad \rightarrow \quad T+4 = X$

Cipher Text is: ACCECX.

Hence, Cipher text for the Harmonious labeling and Sequential labeling will be same for the modified operations case.

Graceful Labelled Tree

From the figure 4, For G(7, 6), 'a' will be from left partite set i.e. {0, 1, 2} and 'b' will be from right partite set i.e. {6, 5, 4, 3}.

(Operation 1): Plain Text is: ACT

$(0\oplus_6 6)\oplus_6(0\odot_6 6) = 0 \quad \rightarrow \quad A+0 = A$

$(1\oplus_6 6)\oplus_6(1\odot_6 6) = 1 \quad \rightarrow \quad C+1 = D$

$(1\oplus_6 5)\oplus_6(1\odot_6 5) = 5 \quad \rightarrow \quad C+5 = H$

$(1\oplus_6 4)\oplus_6(1\odot_6 4) = 3 \quad \rightarrow \quad C+3 = F$

$(1\oplus_6 3)\oplus_6(1\odot_6 3) = 1 \quad \rightarrow \quad C+1 = D$

$(2\oplus_6 3)\oplus_6(2\odot_6 3) = 5 \quad \rightarrow \quad T+5 = Y$

Cipher Text is: ADHFDY.

(Operation 2):

$(0\oplus_6 6)\odot_6(0\odot_6 6) = 0 \quad \rightarrow \quad A+0 = A$

$(1\oplus_6 6)\odot_6(1\odot_6 6) = 0 \quad \rightarrow \quad C+0 = C$

$(1\oplus_6 5)\odot_6(1\odot_6 5) = 0 \quad \rightarrow \quad C+0 = C$

$(1\oplus_6 4)\odot_6(1\odot_6 4) = 2 \quad \rightarrow \quad C+2 = E$

$(1\oplus_6 3)\odot_6(1\odot_6 3) = 0 \quad \rightarrow \quad C+0 = C$

$(2\oplus_6 3)\odot_6(2\odot_6 3) = 0 \quad \rightarrow \quad T+0 = T$

Cipher Text is: ACCECT.

Method 3(Matrix Operations)

$$A = \begin{array}{cc} & \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \boxed{3} & 0 & 0 & 0 \\ \boxed{4} & 0 & 0 & 0 \\ \boxed{5} & \boxed{6} & 0 & 0 \\ 0 & \boxed{0} & \boxed{1} & \boxed{2} \end{bmatrix} \end{array}$$

Matrix (a)

$$B = \begin{array}{cc} & \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \boxed{4} & 0 & 0 & 0 \\ \boxed{5} & 0 & 0 & 0 \\ \boxed{6} & \boxed{0} & 0 & 0 \\ 0 & \boxed{1} & \boxed{2} & \boxed{3} \end{bmatrix} \end{array}$$

Matrix (b)

Matrix (a) : Newly Developed Harmonious Label Matrix; boxes contain induced edge labels

Matrix(b)Newly Developed Felicitous Label Matrix; boxes contain induced edge labels

$$C = \begin{array}{cc} & \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \boxed{3} & 0 & 0 & 0 \\ \boxed{4} & 0 & 0 & 0 \\ \boxed{5} & \boxed{6} & 0 & 0 \\ 0 & \boxed{7} & \boxed{8} & \boxed{9} \end{bmatrix} \end{array}$$

Matrix (c)

$$D = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 7 & 6 & 5 & 4 \end{array} \\ \begin{bmatrix} \boxed{7} & 0 & 0 & 0 \\ \boxed{6} & 0 & 0 & 0 \\ \boxed{5} & \boxed{4} & 0 & 0 \\ 0 & \boxed{3} & \boxed{2} & \boxed{1} \end{bmatrix} \end{array}$$

Matrix (d)

Matrix (c) :Newly Developed Sequential Label Matrix; boxes contain induced edge labels

Matrix (d) : Newly Developed Graceful Label Matrix; boxes contain induced edge labels

We will apply different Matrix operations (i.e. Matrix multiplication, Adjoint of Matrix) on newly developed Matrices (Matrices (a), (b), (c), (d)) for various labelling (Figures5, 6, 7, 8 respectively) and the entries of edge labelings in relavent positions of new obtained matrices can be used to calculate the Cipher text. Examples are given here for illustration.

Encryption: Send the Bipartite Tree (as adjacency matrix), Label Matrix and matrix after the operation to the Receiver.

Decryption: Obtain the Plain text after subtraction of the relevant entries of the operated matrix (say Adjoint of the Label Matrix).

Harmonious Labelled Tree

Plain Text is "ACTS". That is labeled as shown in figure 5.

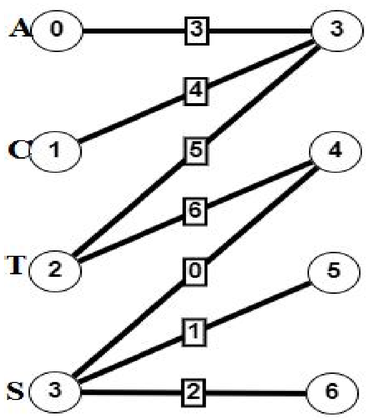

Figure 5: Harmonious Labelled Graph

Newly developed matrix A is shown in Matrix (a). The entries in boxes are edge labels on respective edge connecting related vertices.

(Matrix Multiplication/Square of Label Matrix)

$$A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \boxed{3} & 0 & 0 & 0 \\ \boxed{4} & 0 & 0 & 0 \\ \boxed{5} & \boxed{6} & 0 & 0 \\ 0 & \boxed{0} & \boxed{1} & \boxed{2} \end{bmatrix} \end{array}$$

$$A*A = A^2 = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 \\ 39 & 0 & 0 & 0 \\ 5 & 6 & 2 & 4 \end{bmatrix}$$

Here darkened entries are the newly obtained entries respective to edge label entries.

Hence add these newly induced edge entries to obtain Cipher text.

A+9 = J, C+12 = O, T+39 = G, T+0 = T, S+6 = Y, S+2 = U, S+4 = W.

Cipher Text is: JOGTYUW.

(Adjoint of Matrix A)

$$Adj(A) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 48 & -36 & 0 & 0 \\ -24 & 18 & 0 & 0 \end{bmatrix}$$

Here darkened entries are the newly obtained entries respective to edge label entries.

Hence add these newly induced edge entries to obtain Cipher text.

A+0 = A, C+0 = C, T+48 = P, T-36 = J, S+18 = K, S+0 = S, S+0 = S.

Cipher Text is: ACPJKSS.

Felicitous Labelled Tree

Plain Text is"ACTS". That is labeled as shown in figure 6.

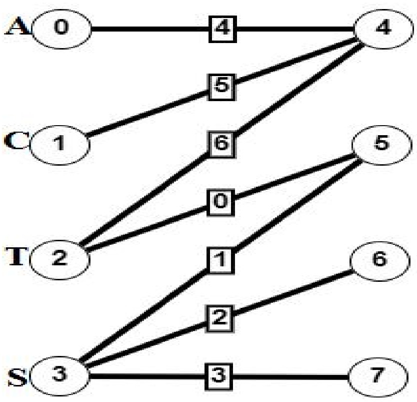

Figure 6: Felicitous Labelled Graph

Newly developed matrix B is shown in Matrix (b). The entries in boxes are edge labels on respective edge connecting related vertices.

(Matrix Multiplication/Square of Label Matrix))

$$B = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} \boxed{4} & 0 & 0 & 0 \\ \boxed{5} & 0 & 0 & 0 \\ \boxed{6} & \boxed{0} & 0 & 0 \\ 0 & \boxed{1} & \boxed{2} & \boxed{3} \end{bmatrix} \end{array}$$

$$B*B = B^2 = \begin{bmatrix} 16 & 0 & 0 & 0 \\ 20 & 0 & 0 & 0 \\ 24 & 0 & 0 & 0 \\ 17 & 3 & 6 & 9 \end{bmatrix}$$

Here darkened entries are the newly obtained entries respective to edge label entries.

Hence add these newly induced edge entries to obtain Cipher text.

A+16 = Q, C+20 = W, T+24 = R, T+0 = T, S+3 = V, S+6 = Y, S+9 = B.

Cipher Text is: QWRTVYB.

(Adjoint of Matrix B)

$$Adj(B) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here darkened entries are the newly obtained entries respective to edge label entries.

Hence add these newly induced edge entries to obtain Cipher text.

A+0 = A, C+0 = C, T+0 = T, T+0 = T, S+0 = S, S+0 = S, S+0 = S.

Cipher Text is: ACTTSSS.

Sequential Labelled Tree

Plain Text is "ACTS". That is labeled as shown in figure 7.

Newly developed matrix C is shown in Matrix (c). The entries in boxes are edge labels on respective edge connecting related vertices.

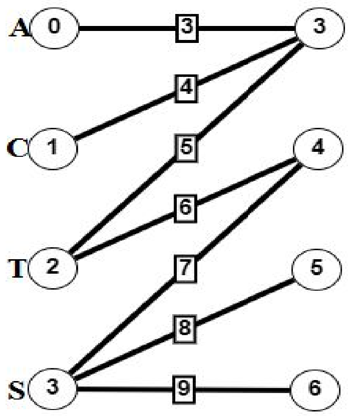

Figure 7: Sequential Labelled Graph

(Matrix Multiplication/Square of Label Matrix)

$$C = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \boxed{3} & 0 & 0 & 0 \\ \boxed{4} & 0 & 0 & 0 \\ \boxed{5} & \boxed{6} & 0 & 0 \\ 0 & \boxed{7} & \boxed{8} & \boxed{9} \end{bmatrix} \end{array}$$

$$C*C = C^2 = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 \\ 39 & 0 & 0 & 0 \\ 68 & 111 & 72 & 81 \end{bmatrix}$$

Here darkened entries are the newly obtained entries respective to edge label entries.

Hence add these newly induced edge entries to obtain Cipher text.

A+9 = J, C+12 = O, T+39 = G, T+0 = T, S+111 = Z, S+72 = M, S+81 = V.

Cipher Text is: JOGTZMV.

(Adjoint of Matrix C)

$$Adj(C) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 216 & -162 & 0 & 0 \\ -192 & 144 & 0 & 0 \end{bmatrix}$$

Here darkened entries are the newly obtained entries respective to edge label entries.

Hence add these newly induced edge entries to obtain Cipher text.

A+0 = A, C+0 = C, T+216 = B, T-162 = N, S+144 = G, S+0 = S, S+0 = S.

Cipher Text is: ACBNGSS.

Graceful Labelled Tree

Plain Text is "ACTS". That is labeled as shown in figure 8.

Newly developed matrix D is shown in Matrix (d). The entries in boxes are edge labels on respective edge connecting related vertices.

$$D = \begin{array}{cc} & \begin{array}{cccc} 7 & 6 & 5 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \boxed{7} & 0 & 0 & 0 \\ \boxed{6} & 0 & 0 & 0 \\ \boxed{5} & \boxed{4} & 0 & 0 \\ 0 & \boxed{3} & \boxed{2} & \boxed{1} \end{bmatrix} \end{array}$$

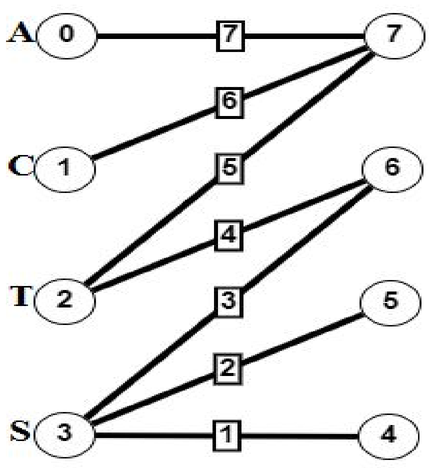

Figure 8: Graceful Labelled Graph

(Matrix Multiplication/Square of Label Matrix))

$$D*D = D^2 = \begin{bmatrix} 49 & 0 & 0 & 0 \\ 42 & 0 & 0 & 0 \\ 59 & 0 & 0 & 0 \\ 28 & 11 & 2 & 1 \end{bmatrix}$$

Here darkened entries are new obtained entries respective to edge label entries.

Hence add these newly induced edge entries to obtain Cipher text.

A+49 = X, C+42 = S, T+59 = A, T+0 = T, S+11 = D, S+2 = U, S+1 = T.

Cipher Text is: XSATDUT.

(Adjoint of Matrix D)

$$Adj(D) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 24 & -28 & 0 & 0 \\ -48 & 56 & 0 & 0 \end{bmatrix}$$

Here darkened entries are the newly obtained entries respective to edge label entries.

Hence add these newly induced edge entries to obtain Cipher text.

A+0 = A, C+0 = C, T+24 = R, T-28 = R, S+56 = W, S+0 = S, S+0 = S.

Cipher Text is: ACRRWSS.

Method 4 (Matrix Transformation/Change in Basis):

Plain Text is: ACTS

Harmonious Labelling

A is the newly developed Harmonious Label Matrix as shown in Matrix (a)..Linear Transformation (L.T.) is being used here.

Encryption: Send the L.T. , Label Matrix and relevant information regarding calculation of $N = U^{-1}SU$ and Cipher text to the Receiver.

Decryption: Depending on the Label Matrix and other relevant information, compute the Plain text by subtraction.

Illustration:

Consider an L.T. $L : \mathbb{R}^4 \to \mathbb{R}^4$ s.t. $L\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 5 & 6 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$.

Then, to find matrix of L w.r.t. basis $u = (u_1, u_2, u_3, u_4)$. Where u_1 = (1, 1, 1, -1), u_2 = (1, 1, -1, 1), u_3 = (1, -1, 1, 1) and u_4 = (-1, 1, 1, 1).

Let 'S' be the matrix of L w.r.t. standard basis (e_1, e_2, e_3, e_4). So, S = A. The standard basis is as follows:

e_1 = (1,0,0,0,), e_2=(0,1,0,0), e_3= (0,0,1,0),, e_4= (0,0,0,1)

'N' be the matrix of L w.r.t. basis $u = (u_1, u_2, u_3, u_4)$.

'U' be the transition matrix from u_1, u_2, u_3, u_4 to e_1, e_2, e_3, e_4.

$$U = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$$

Hence, $N = U^{-1}SU$

$$N = \frac{1}{4}\begin{bmatrix} 19 & 17 & 3 & -9 \\ -5 & -3 & 11 & -5 \\ 9 & 11 & 1 & 5 \\ 11 & 13 & 3 & 3 \end{bmatrix}$$

Discard ¼ common in N and replace the darkened entries from respective edge labelling to obtain the Cipher text.

A+19 = T, C-5 = X, T+9 = C, T+11 = E, S+13 = F, S+3 = V, S+3 = V.

Cipher Text is: TXCEFVV.

Felicitous Labelling

B is the newly developed Felicitous Label Matrix as shown in Matrix (b).

Illustration:

Consider an L.T. $L : \mathbb{R}^4 \to \mathbb{R}^4$ s.t. $L\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = B\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$.

Then, to find matrix of L w.r.t. basis $u = (u_1, u_2, u_3, u_4)$. Where u_1 = (1, 1, 1, -1), u_2 = (1, 1, -1, 1), u_3 = (1, -1, 1, 1) and u_4 = (-1, 1, 1, 1).

Let 'S' be the matrix of L w.r.t. standard basis (e_1, e_2, e_3, e_4). So, S = B.

'N' be the matrix of L w.r.t. basis $u = (u_1, u_2, u_3, u_4)$.

'U' be the transition matrix from u_1, u_2, u_3, u_4 to e_1, e_2, e_3, e_4.

$$U = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$$

Hence, $N = U^{-1}SU$

$$N = \frac{1}{4}\begin{bmatrix} 15 & 13 & 11 & -21 \\ 3 & 5 & 7 & 3 \\ 5 & 7 & 9 & 1 \\ 7 & 9 & 11 & -1 \end{bmatrix}$$

Discard ¼ common in N and replace the darkened entries from respective edge labelling to obtain the Cipher text.

A+15 = P, C+3 = F, T+5 = Y, T+7 = A, S+9 = B, S+11 = D, S-1 = R.

Cipher Text is: PFYABDR.

Sequential Labelling

C is the newly developed Sequential Label Matrix as shown in Matrix (c).

Illustration:

Consider an L.T. $L : \mathbb{R}^4 \to \mathbb{R}^4$ s.t. $L\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = C\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 5 & 6 & 0 & 0 \\ 0 & 7 & 8 & 9 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$.

Then, to find matrix of L w.r.t. basis $u = (u_1, u_2, u_3, u_4)$. Where u_1 = (1, 1, 1, -1), u_2 = (1, 1, -1, 1), u_3 = (1, -1, 1, 1) and u_4 = (-1, 1, 1, 1).

Let 'S' be the matrix of L w.r.t. standard basis (e_1, e_2, e_3, e_4). So, S = C.

'N' be the matrix of L w.r.t. basis $u = (u_1, u_2, u_3, u_4)$.

'U' be the transition matrix from u_1, u_2, u_3, u_4 to e_1, e_2, e_3, e_4.

$$U = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$$

Hence, $N = U^{-1}SU$

$$N = \frac{1}{4}\begin{bmatrix} 12 & 10 & -4 & -30 \\ 2 & 4 & 18 & 16 \\ 16 & 18 & 8 & 26 \\ 18 & 20 & 10 & 24 \end{bmatrix}$$

Discard ¼ common in N and replace the darkened entries from respective edge labelling to obtain the Cipher text.

A+12 = M, C+2 = E, T+16 = J, T+18 = L, S+20 = M, S+10 = C, S+24 = Q.

Cipher Text is: MEJLMCQ.

Graceful Labelling

D is the newly developed Sequential Label Matrix as shown in Matrix (d).

Illustration:

Consider an L.T. $L : \mathbb{R}^4 \to \mathbb{R}^4$ s.t. $L\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = D\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} =$

$$\begin{bmatrix} 7 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 5 & 4 & 0 & 0 \\ 0 & 3 & 2 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}.$$

Then, to find matrix of L w.r.t. basis $u = (u_1, u_2, u_3, u_4)$. Where u_1= (1, 1, 1, -1), u_2 = (1, 1, -1, 1), u_3 = (1, -1, 1, 1) and u_4= (-1, 1, 1, 1).

Let 'S' be the matrix of L w.r.t. standard basis (e_1, e_2, e_3, e_4). So, S = D.

'N' be the matrix of L w.r.t. basis $u = (u_1, u_2, u_3, u_4)$.

'U' be the transition matrix from u_1, u_2, u_3, u_4 to e_1, e_2, e_3, e_4.

$$U = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$$

Hence, $N = U^{-1}SU$

$$N = \frac{1}{4}\begin{bmatrix} 18 & 20 & 14 & -20 \\ 8 & 6 & 12 & -6 \\ 14 & 12 & 2 & 4 \\ 12 & 10 & 0 & 6 \end{bmatrix}$$

Discard ¼ common in N and replace the darkened entries from respective edge labelling to obtain the Cipher text.

A+18 = S, C+8 = K, T+14 = H, T+12 = F, S+10 = C, S+0 = S, S+6 = Y.

Cipher Text is: SKHFCSY.

CONCLUSION

In this work Bipartite Trees labelled with four kinds of vertex labelling schemes namely harmonious, felicitous, sequential and gracefulfor both unequal and equal partite sets have been presented for application in Cryptography. Methods involving adding of vertex labels and induced edge labels, two kinds of modified modulo operations, Matrix operations such as Adjoint, matrix multiplication such as squaring of a matrix, Matrix Transformation/Change of Basis for an enhanced security to hide the Cipher text even more, have been developed. Varied and numerous complex Cipher texts are obtained with multiple layers of hiding making them suitable for a desirable and safe data transfer. Apart from these methods some more suggestions are as follows :

1. Operations such as matrix multiplication of higher powers such as A^3, A^4 etc. also can be employed.
2. More kinds of modulo arithmetic with various combinations of modified modulo operations can be tried for still more safe data transfer.
3. In case of unequal partite sets, more number of vertices can be taken in the right partite set so as to increase the number of characters in the Cipher text to make it even more unintelligible to the unwanted party.
4. In the Method 4 of matrix transformation/change of basis, instead of ordinary addition to compute the Cipher texts, modified modulo operations also can be taken in order to achieve a still greater hiding and complexity of the Cipher texts. Therefore numerous, complex and much hidden Cipher texts offering high secrecy and privacy for a much safer and desirable data transfer can be obtained based on the labelled Bipartite Trees.

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