Indian J.Sci.Res. 16 (1): 112-119, 2017 ISSN: 0976-2876 (Print)
ISSN: 2250-0138(Online)

STABILITY ANALYSIS OF LOGISTIC GROWTH MODEL WITH IMMIGRATION EFFECT

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ABSTRACT

In the present paper emphasized with stability analysis of logistic growth model with immigration effects. In this paper have characterized by the modified logistic growth equation $\frac{dP}{dt} = aP(1-\frac{P}{K}) + f(P)$, where P is the population size at time t, K is the carrying capacity and a (constant) is the natural birth rate of the population. The function $f(P)$ represents the characteristic of immigration. The presence of immigrants affects the birth rates and carrying capacity of the population. We analyzed the stability of equilibrium points of logistic growth model and modified logistic growth model due to immigration effects when the immigration function $f(P)$ is a constant and proportional to P^{α} $(\alpha = 1,2,3)$. Equilibrium points are identifies and effective birthrate, carrying capacity are computed in each cases. We established the criteria of stability and all trajectories (Population vs. Time) are clarified. In this paper we have studied about the logistic growth model to simplifying with the solutionof the two equilibrium points and geometrical analysisthrough mathematical models. Further we modifiedlogistic equation with Immigration effect and discussed about four special cases of the model, to use small perturbation around equilibrium point and to take stability that the real part of the equilibrium state that is stable, unstable and neutrally stable.

Keywords: carrying capacity, equilibrium points, Immigration, logistic growth model, stability

Mathematical modeling as a tool in the study of population dynamics has a long and diverse history, spanning at least three centuries. Although a multitude of models has been put forward and many challenging problems have been solved, the endeavor to explain and formulate general principles underlying the dynamics of populations in space and time is far from over. Basically a population dynamics model answers the question how a population is going to change in the (near) future, its current status and the environmental condition that the population exposes to. These simplified population models usually start with four key variables including death, birth, immigration and emigration. Mathematical methods have been used to model population dynamics since the twelfth century. Studies on single species population growth were initiated by (Malthus, 1798) and proposed a simple exponential growth model in his work 'An Essay on the Principle of Population'. He developed a mathematical model hypothesizing that human populations have a constant natural growth rate. If $P(t)$ represent the population size at time t then *Malthus Growth Model* is $\frac{dP}{dt} = aP$, the solution of which $P = P_0 e^{at}$ where P_0 is the initial population and a is the growth constant. His model was subsequently modified by other researchers taking into account several self-limiting features of populations. The Logistic growth model was proposed by(Verhulst, 1845). His model incorporated the idea of carrying capacity. Thus the population growth is not only on how to depend on the population size; but also on how far this size is from its upper limit i.e. its carrying capacity. He modified Malthus's model to make a population size proportional to both the previous population and a new term $\frac{a-bP(t)}{a}$. Where a and b are vital coefficients of the population. So the modified equation using this new term is: $\frac{dP}{dt} = aP(1-\frac{P}{K})$ where $K = \frac{a}{b}$ is the carrying capacity and b is a constant.Population dynamics, especially the equilibrium states and their stability, have traditionally been analyzed using mathematical models. Of interest in population models are the

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equilibrium states and convergence towards these states.

LOGISTIC GROWTH MODEL

The logistic equation anticipates a limit or carrying capacity K, on population growth..
$\frac{dP}{dt} = aP(1-\frac{P}{K}),\ P_o = P(0)$ at t = 0 (1)

Solving for $P(t)$ and simplifying further provides us with the solution

$P(t) = \frac{P_o K}{P_o + (K - P_o)e^{-at}}$ (2)

Plotting the population' growth as a function of time shows P approaching K along a sigmoid (S-Shaped) curve when the population's initial state P_0 is below $K/2$; above $K/2$

Obtaining Equilibrium Points

We obtain the system's equilibrium points P^* by finding all values of *P* that satisfy

$\frac{dP}{dt} = 0 \Rightarrow aP(1-\frac{P}{K}) = 0$ (3)

And we have exactly two equilibrium points $P^* = 0$ and $P^* = K$.

GEOMETRICAL ANALYSIS

The trivial equilibrium point $P^* = 0$ is unstable, and the second equilibrium point $P^* = K$ represents the stable equilibrium, where P asymptotically approaches the carrying capacity *K*. In terms of limit we can say $\lim_{t\to\infty} P(t) = \frac{a}{b} = K,\ P(0) > 0$. A point of inflection occurs at $P = K/2$ for all solutions that cross it and we can see graphically that growth of *P* is rapid until it passes the inflection point. From there subsequent growth shows as *P* asymptotically approaches *K*. If $P > K$ then $dP/dt < 0$ and *P* decreases exponentially towards *K*. This case only occurs when the initial condition $P(0) = P_0 > K$.

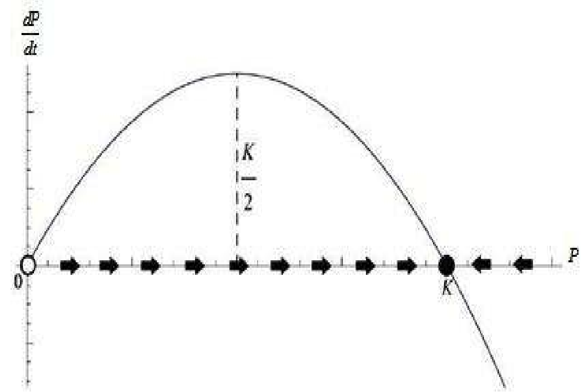

Figure 1: Phase line portrait of logistic growth

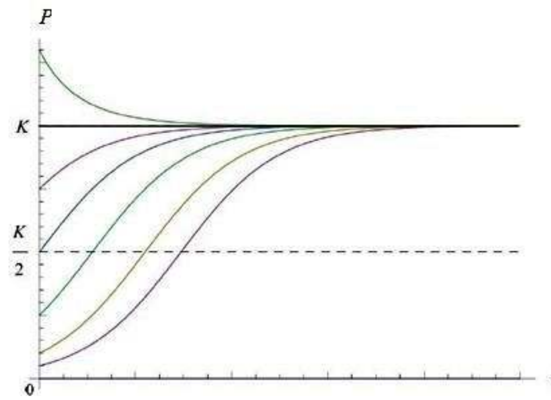

Figure 2: Dynamics of the logistic model

For more quantitative measure of system's stability, we may linearize the equation in the neighborhood of its equilibrium point. Let $P(t) = P^* + \varepsilon(t)$, where $\varepsilon(t)$ a small perturbation in the neighborhood of an equilibrium point denoted is P^*. We are interested in whether the perturbation grows or decay, so consider

$\frac{d\varepsilon}{dt} = \frac{d}{dt}(P - P^*) \Rightarrow \frac{dP}{dt} = f(P) = f(P^* + \varepsilon)$ (4)

Performing Taylor series expansion on equation (4) yields

$f(P^* + \varepsilon) = f(P^*) + \varepsilon \left.\frac{df}{dP}\right|_{P=P^*} + ...,$ (5)

Where ellipsis denotes quadratic ally small non linear terms in ε. We may also estimate the term $f(P^*)$ since it is equal to zero, and we are provided with the approximatedequation

$f(P^* + \varepsilon) \approx \varepsilon \left.\frac{df}{dP}\right|_{P=P^*}$ (6)

Thus $f(P) = aP(1-\frac{P}{K}) = aP - \frac{aP^2}{K}$ and $\frac{df(P)}{dP} = a - \frac{2aP}{K}$ (7)

Hence near the equilibrium points $P^* = 0$ and $P^* = K$, we obtain

$$\frac{d\varepsilon}{dt} \approx \varepsilon(a - \frac{2aP}{K})\Big|_{P=0} = a\varepsilon \quad (8)$$

$$\frac{d\varepsilon}{dt} \approx \varepsilon(a - \frac{2aP}{K})\Big|_{P=K} = \varepsilon(a-2a) = -a\varepsilon \quad (9)$$

We see that $\frac{d\varepsilon}{dt} \approx \pm a\varepsilon$ takes its form. Since $a > 0$, these results indicate that the equilibrium point $P^* = 0$ is unstable since the perturbation $\varepsilon(t)$ grows exponentially if $f'(P^*) > 0$. On the other hand, $P^* = K$ is stable since $\varepsilon(t)$ decays exponentially if $f'(P^*) < 0$. Additionally the magnitude of $f'(P^*)$ tells how rapidly exponential growth or decay will occur, and its reciprocal $\left|f'(P^*)\right|^{-1}$ is called characteristic time scale which gives the amount of time it takes for $P(t)$ to vary significantly in the neighborhood of P. In this case, the characteristic time scale is $\left|f'(P^*)\right|^{-1} = a^{-1}$ for both equilibrium points.

Logistic Growth Equation with Immigration

Let $f(P)$ be the immigration function. Then modified logistic growth equation is given by

$$\frac{dP}{dt} = aP(1-\frac{P}{K}) + f(P)$$
(10)

We obtain the system's equilibrium point P^* by finding all values of *P* that satisfy

$$\frac{dP}{dt} = 0 \Rightarrow aP(1-\frac{P}{K}) + f(P) = 0 \quad (11)$$

Let P^* be an equilibrium point i.e.

$$aP^*(1-\frac{P^*}{K}) + f(P^*) = 0 \quad (12)$$

To examine the stability of equilibrium point, consider a small perturbation around equilibrium point $P(t) = P^* + \varepsilon(t)$, where ε is a first order small quantity. Using equation (10), we get after neglecting higher power of ε the differential equation for ε

$$\frac{d\varepsilon}{dt} = [bK - 2bP^* + f'(P^*)]\varepsilon \quad (13)$$

$$\Rightarrow \varepsilon = \varepsilon_0 e^{\lambda t} \quad (14)$$

Where ε_0 is the initial value perturbation and $\lambda = [bK - 2bP^* + f'(P^*)]$. When λ is real the equilibrium state is:

Stable if $\lambda < 0 \Rightarrow f'(P^*) < b(2P^* - K)$

Unstable if $\lambda > 0 \Rightarrow f'(P^*) > b(2P^* - K)$

Neutrally stable if $\lambda = 0 \Rightarrow f'(P^*) = b(2P^* - K)$

When λ a complex then criterion for the stability that $real$ part of $\lambda < 0$.

Special cases for the immigration function $f(P)$

Case A. $f(P) = c$ (constant)

Modified Logistic equation is $\frac{dP}{dt} = aP(1-\frac{P}{K}) + c$
(15)

The equilibrium point are given by

$$aP(1-\frac{P}{K}) + c = 0$$

There exists only one equilibrium point

$$P^* = \frac{bK + \sqrt{(bK)^2 - 4cb}}{2b} (>0)$$

The linearized equation for ε is

$$\frac{d\varepsilon}{dt} = [bk - 2b(\frac{bK + \sqrt{(bK)^2 - 4cb}}{2b})\varepsilon \Rightarrow \varepsilon = \varepsilon_0 e^{-\lambda t}$$
(16)

Where $\lambda = \sqrt{(bK)^2 + 4cb} > 0$, as $t \to \infty, \varepsilon \to 0$, hence equilibrium state is stable.

Now the solution of (15)is given by

$$P = \frac{K}{2} + v\left\{\frac{1 + ue^{-2bvt}}{1 - ue^{-2bvt}}\right\} \quad (17)$$

Where

$$u = \frac{2b(P_0 - v) - a}{2b(P_0 + v) - a} \text{ and } v = \sqrt{\frac{a^2 + 4bc}{4b^2}} = \sqrt{\frac{K^2}{4} + \frac{c}{b}}$$

As $t \to \infty, P(t) \to \frac{K}{2} + \sqrt{\frac{K^2}{4} + \frac{c}{b}}$ is the effective carrying capacity and which is more than natural carrying capacity K.

Trajectories (Population vs. Time):

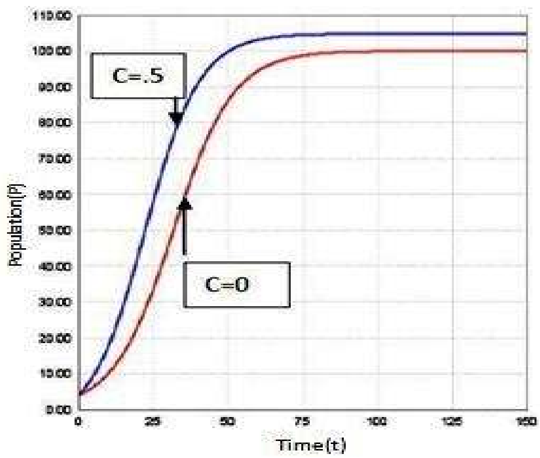

Figure 3 : $P_0 < K/2, P_0 = 5, K = 100, b = .001$

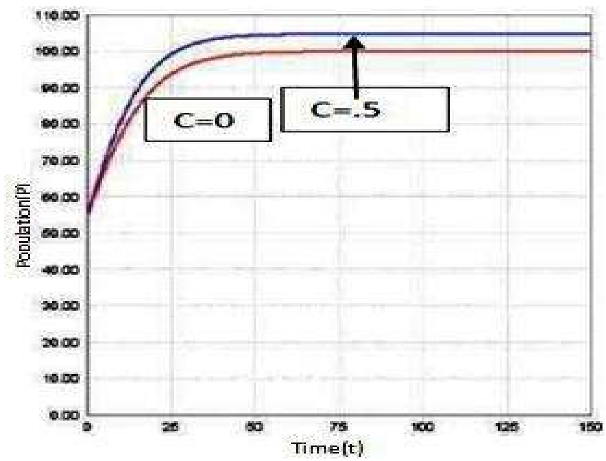

Figure 4 : $P_0 > K/2, P_0 = 55, K = 100, b = .001$

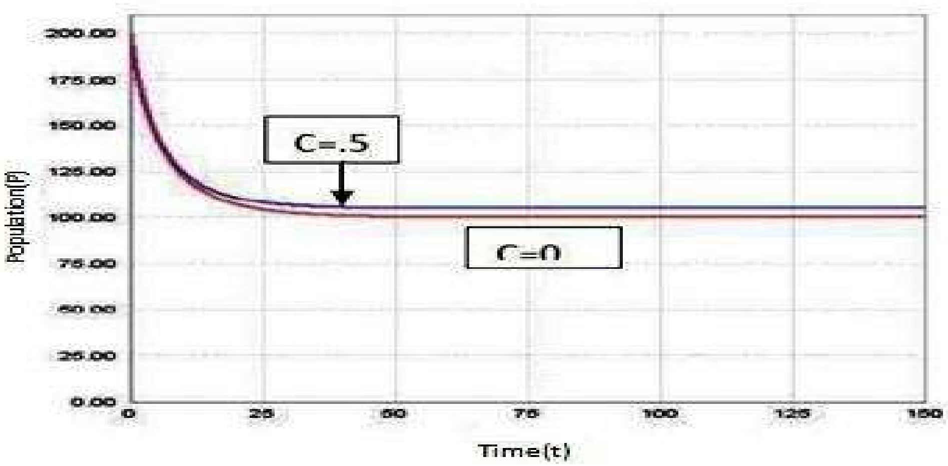

Figure 5 : $P_0 > K, P_0 = 200, K = 100, b = .001$

Case B: $f(P) \propto P$ i.e. $f(P) = cP$

Then the modified logistic equation is

$$\frac{dP}{dt} = aP(1 - \frac{P}{K}) + cP = bP[(K + \frac{c}{b}) - P] \quad (18)$$

The equilibrium point are given by

$$bP[(K + \frac{c}{b}) - P] = 0$$

Here we find two equilibrium points $P_1^* = 0$ and $P_2^* = \frac{a+c}{b}$

For $P_1^* = 0$, the linearized equation for ε is $\frac{d\varepsilon}{dt} = bK\varepsilon \Rightarrow \varepsilon = \varepsilon_0 e^{at}$, $a > 0$ .As $t \to \infty, \varepsilon \to \infty$ .Hence $P_1^* = 0$ is unstable.For $P_2^* = \frac{a+c}{b}$, the linearized equation for ε is $\frac{d\varepsilon}{dt} = -(a+2c)\varepsilon \Rightarrow \varepsilon = \varepsilon_0 e^{-(a+2c)t}$

As $t \to \infty, \varepsilon \to 0$. Hence $P_2^* = \frac{a+c}{b}$ is stable. Equation (18) is same as the logistic model with increased natural birth rate and carrying capacity is increase from K to $K+\frac{c}{b}$.

Solution of equation (18) is given by

$$P = \frac{(K + \frac{c}{b})P_0}{P_0 + (K + \frac{c}{b} - P_0)e^{-Kbt}}$$

(19)

Trajectories (Population vs. Time):

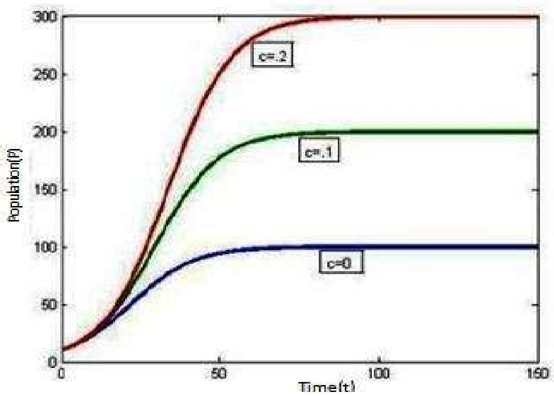

Figure 7: $P_0 > K/2, P_0 = 60, K = 100, b = .001$

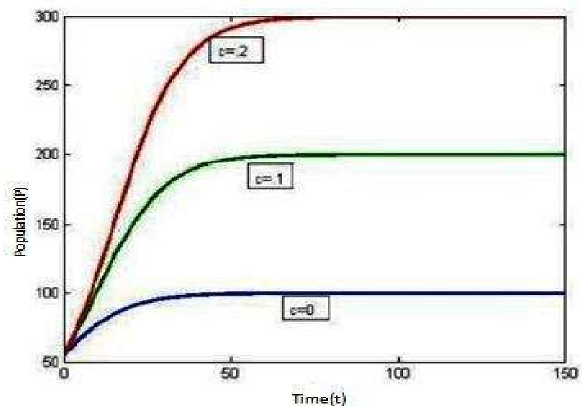

Figure 6: $P_0 < K/2, P_0 = 10, K = 100, b = .001$

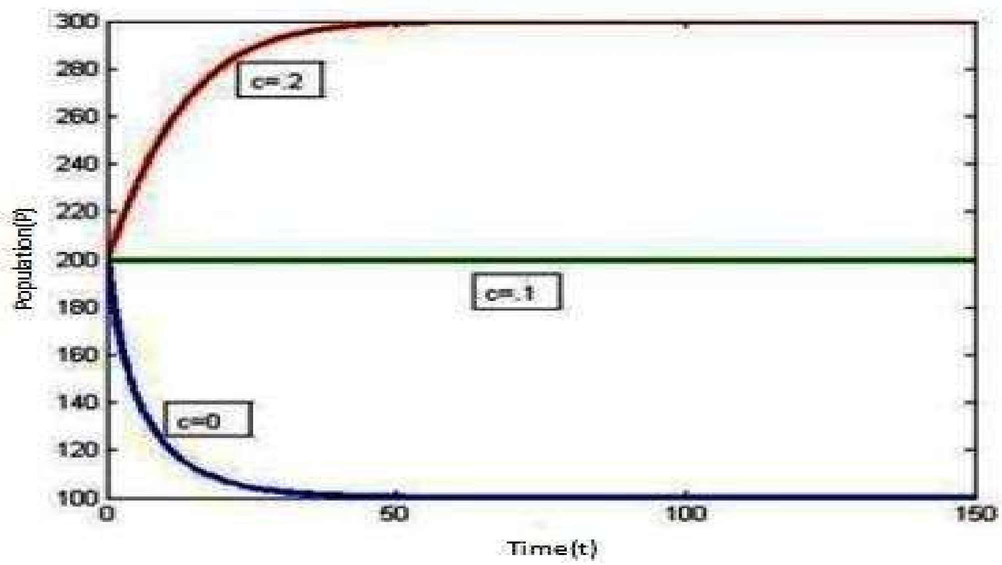

Figure 8: $P_0 > K, P_0 = 200, K = 100, b = .001$

Case C: $f(P) \propto P^2$ **i.e.** $f(P) = cP^2$

Then the growth rate equation is

$$\frac{dP}{dt} = aP(1-\frac{P}{K}) + cP^2 = bP[(K-(1-\frac{c}{b})P]$$

(20)

This equation is same as the logistic model equation with increased carrying capacity from K to $\frac{K}{1-c/b}$ without change in natural birthrate. Equation (20) has two equilibriumpoints P_1^*=0 and $P_2^* = \frac{a}{b-c}$ where

P_1^* is unstable and P_2^* is stable .

Solution of equation (20) is

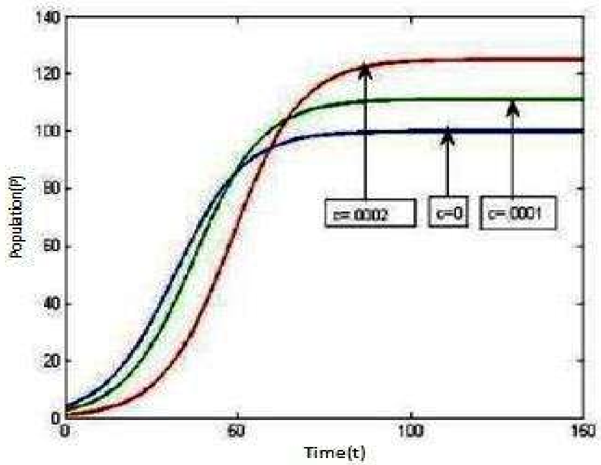

Figure 10: $P_0 > K/2, P_0 = 60, K = 100, b = .001$

$$P = \frac{\left(\frac{bK}{b-c}\right)P_0}{P_0 + \left(\frac{bK}{b-c} - P_0\right)e^{-Kbt}}, (b > c)$$

Trajectories (Population vs. Time):

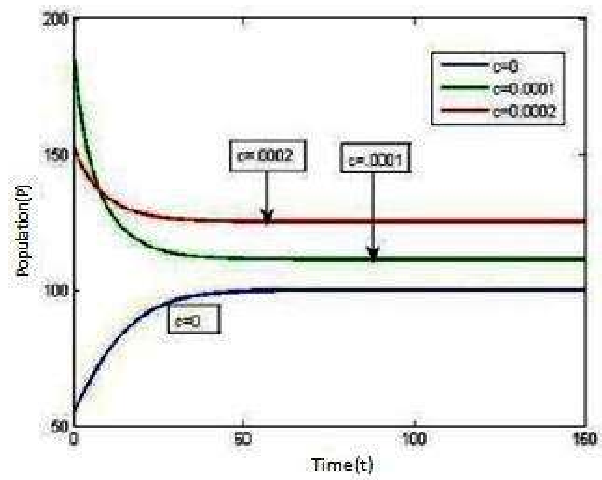

Figure 9: $P_0 < K/2, P_0 = 6, K = 100, b = .001$

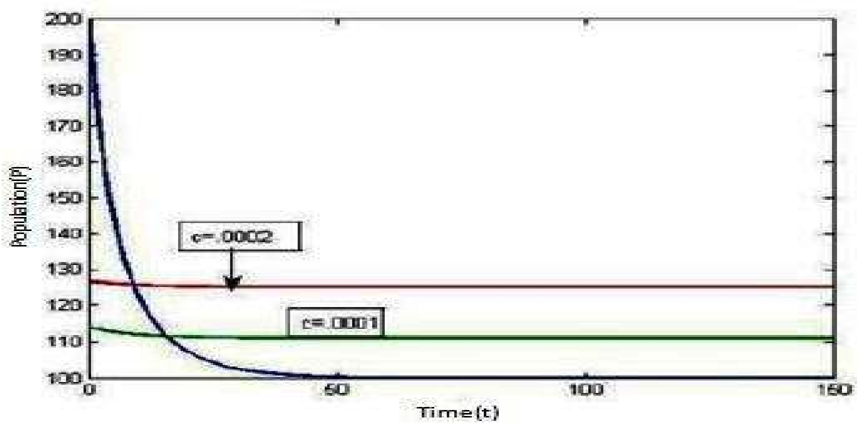

Figure 11: $P_0 > K, P_0 = 200, K = 100, b = .001$

Case D: $f(P) \propto P^3$ i.e. $f(P) = cP^3$

Growth rate equation is

$$\frac{dP}{dt} = aP(1 - \frac{P}{K}) + cP^3 = bP[(K-P) + \frac{c}{b}P^2]$$

(22)

The equilibrium states are given by

$$\frac{dP}{dt} = 0 \Rightarrow bP[(K - (1 - \frac{c}{b})P] = 0 \quad (23)$$

The roots of equation (23) are

$$P_1^* = 0, P_2^* = \frac{b + \sqrt{b^2 - 4cbK}}{2c} \text{ and } P_3^* = \frac{b - \sqrt{b^2 - 4cbK}}{2c}$$

Using a small perturbation in ε, $P = P^* + \varepsilon$ then we get the linearized equation for ε

$$\frac{d\varepsilon}{dt} = [bK - 2bP^* + 3cP^{*2}]\varepsilon$$

(24)

Now here three sub cases arise-

(i) $\boldsymbol{b} < 4\boldsymbol{cK}$, then there exists only one equilibrium point $P_1^* = 0$. As before the equilibrium point $P_1^* = 0$ is unstable.

(ii) $\boldsymbol{b} = 4\boldsymbol{cK}$, then there will be two equilibrium points $P_1^* = 0$ and $P_2^* = \frac{b}{2c}$. As before $P_1^* = 0$ is unstable and for the stability of $P_2^* = \frac{b}{2c}$, the linearized equation for ε is given by

$$\frac{d\varepsilon}{dt} = \left(bK - \frac{b^2}{4c}\right)\varepsilon \Rightarrow \varepsilon = \varepsilon_0 e^{\left(bK - \frac{b^2}{4c}\right)t} \quad (25)$$

Equilibrium state is:

Stable if $c < b/4K$

Unstable if $c > b/4K$

Neutrally stable if $c = b/4K$

(iii) $\boldsymbol{b} > 4\boldsymbol{cK}$, then there exists three equilibrium points

$$P_1^* = 0, P_2^* = \frac{b + \sqrt{b^2 - 4cbK}}{2c} \text{ and } P_3^* = \frac{b - \sqrt{b^2 - 4cbK}}{2c}$$

. As before P_1^* is unstable. Let $P = P_2^* + \varepsilon$ is a small perturbation for P_2^* then linearized equation is

$$\frac{d\varepsilon}{dt} = \left[bK - 2b\left(\frac{b + \sqrt{b^2 - 4cbK}}{2c}\right) + 3c\left(\frac{b + \sqrt{b^2 - 4cbK}}{2c}\right)\right]\varepsilon$$

Let $\rho = \sqrt{b^2 - 4cbK}$ then

$$\varepsilon = \varepsilon_0 e^{-(-2bK - b^2/4c - b-\rho/2c)t} \quad (26)$$

Equilibrium state P_2^* is

Stable if $c > b/4K$

Unstable if $c \le b/4K$

For P_3^* the linearized equation is

$$\frac{d\varepsilon}{dt} = \left[bK - 2b\left(\frac{b - \sqrt{b^2 - 4cbK}}{2c}\right) + 3c\left(\frac{b - \sqrt{b^2 - 4cbK}}{2c}\right)\right]\varepsilon$$

Then $\varepsilon = \varepsilon_0 e^{-(-2bK - b^2/4c + b-\rho/2c)t}$ (27)

Equilibrium state P_3^* is

Stable if $c \ge b/4K$

Unstable if $c < b/4K$

DISCUSSION

The analysis of Logistic growth population model fromequation (1) ha quantitative measure of system's stability we use a small perturbation in the neighborhood of equilibrium point P^* and obtained equations(8) and (9), where results are same as above $P^* = 0$ unstable and $P^* = K$ is stable. Secondly addedan immigration characteristic $f(P)$ into classical logistic growth equation and analyze the stability of equilibrium states of modified logistic equation (10) in four special cases .The birthrate and carrying capacity of population are affected by the immigrants. In each of the cases the effective birth rate, carrying capacity are computed, equilibrium points are identified and criteria of the stability established. All the trajectoriesshow that the carrying capacity is increased with a small variation of $'c'$. When the immigration rate is proportional to the square of population P then a very small value of $'c'$ is identified. Unchecked immigration into countries may lead to overpopulation to the point where those countries no longer have the required resources for their population. This is particularly problematic in countries where immigration numbers far exceed emigration numbers. In some cases, immigrants may be attempting to escape overpopulation in their own countries, only to contribute to the same issues in the countries they move to. However, data also exists to show the immigration can bolster economies to contain many problems.

REFERENCES

Freedman, H.I., 1980. Deterministic Mathematical Models in population Ecology Marcel-Decker, New York.

Glen Ledder, 2005. Differential Equation: A modeling Approach. McGraw-Hill Companies Inc. USA.

J.N Kapur, Q.J Khan., 1979. Some Mathematical Model for Population Growth. Indian

Journal for Pure Applied Math,**10** (3): 277-286.

Malthus T.R., 1798: An Essay on the Principle of Population. J Johnson, London.

Meyer, P. S. and Ausubel, J. H., 2001,Human Population Dynamics with Logistic Model: How Much Can Be Modelled and Predicted. **52**: 1-30.

McCallum H I., 1992: Effects of immigration on chaotic population dynamics; J. Theor. Biol.**154**: 277–284.

Svirezhev andD.O.logofet., 1983: Stability of biological communities, MIR publishers,Moscow.

Kumar, V. andLal, S., 2016: Mathematical Modeling of Population Growth of India, Aryabhatta Journal of Mathematics and Informatics .8 (2):47-56.

Verhulst, P. F.,1845.RechercheMathematiques Sur Le LoiD'accroissement De La Population", Nouveau Memoires De L'Academie Royale Des Sciences ET Belles Letters De Brssuxelles.183-38.