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THERMAL STRESSES FIELD DUE TO A SYSTEM OF GRIFFITH CRACKS LYING AT THE INTERFACE OF TWO BONDED DISSIMILAR MICROPOLAR ELASTIC HALF PLANES

RAJENDRA KUMAR TRIPATHI[1]

Department of Mathematics, Pioneer Mahila Mahavidlaya, Barabanki, U.P., India

ABSTRACT

In bonding two materials with different mechanical elastic properties, very often it is not possible to obtain a homogeneous perfect bond due to the existence of entrapped imperfections as in the joints involving ceramics and metal used in manufacturing electronic devices and variety of reinforced composites. The cavities and other imperfection with weak bond strength existing on the interface usually have very sharp corners. For the purpose of analysis, these imperfections may all be classified as singular surfaces across which displacement or stress vector suffer a discontinuity. These singularities correspond regions of high stress in which fracture of material may occur. Many problems have been solved involving one or more cracks in an infinite elastic medium. The solution is given in terms of a single Fredholm integral euqation then Lowengrub and Srivastava treated in an infinite elastic strip containing a pair of equal size collinear cracks. They used a finite Hilbert transform technique developed by Srivastava and Lowengrub, in order to reduce the problem to a Fredholm integral equation. The problem is reduced to singular integral equations and the stress intensity factors, crack displacement and crack energies are then determined. The stress and displacement field in the vicinity of a Griffith crack located at the interface of two bonded dissimilar elastic half planes are determined. A systematic use of Fourier transforms reduces the problem to that of solving a set of simultaneous dual integral equations which are equivalent to Riemann boundary value problem.

KEYWORDS: Simultaneous Dual and Triple Integral Equations, Riemann Boundary Value Problem Riemann Hilbert Problem and Fredholm Integral Equation of Second Kind

The stress field in the vicinity of a pair of Griffith cracks located at the interface of two bonded dissimilar elastic half-planes is determined in (Srivastava *et. al.*, 2000) Fourier transform is employed in order to reduce the problem to that of solving simultaneous set of triple equations containing a trigonometric kernel.

In the linear theory of micropolar elasticity the problem of Griffith crack in a transverse field of constant uniaxial tension is studied in (Gerasoulis and Srivastav, 1980). The problem is reduced to three Fredholm integral equations of the second kind which have the same kernel are solved numerically. Another problem of a Griffith crack at the interface of two bonded dissimilar micropolar elastic half-planes is considered by (Green and Zerna, 1960). The deformation in two half-planes is due to the application of constant pressure to the faces of the crack. The analysis is carried out by (Lowengrub and Srivastava, 1968) to a system of simultaneous dual integral equations which are further reduced to the system of Riemann-Hilbert problem.

The object of this chapter is to present a general formulation when system of m Grifith cracks are presented at the interface of two bonded dissimilar micropolar elastic half-planes. The cracks are situated with respect to y-axis taken perpendicular to the interface. The problem is first reduced to a system of simultaneous dual integral equations which are further reduced to the solution of Riemann-Hilbert problem. Further the expressions for evaluating the stress intensity factors at the tip of cracks are derived. The calculations have been done in case when constant temperature is prescribed on crack surfaces.

FORMULATION OF THE PROBLEM

We shall study the distribution of thermal stress in the vicinity of a system of a Griffith cracks located at the interface of two bonded dissimilar micropolar elastic half planes which is symmetrically situated with y-axis and perpendicular to the interface. We assume that the two half-planes $y > 0$ and $y < 0$ be occupied by elastic materials with constant μ_i and k_i respectively, μ_i denotes the rigidity modulus and $k_i = 3-4\eta_i$ where η_i denotes the poison ratio of the two elastic materials. The cracks are gien by $y = 0$, $a_j \leq | x | \leq b_j$, $J = 1, 2, 3.....n$ where $0 \leq a_1 < b_1 < a_2 < b_2 .... a_n$ M< b_n.

In case when $a_1 = 0$ then the number of cracks is odd and $m = 2n-1$. If $a_1 \neq 0$, then the number of cracks is even and hence $m = 2n$, in this case, the cracks are given by $y = 0$, $-b_1 \leq x \leq b_1$, $a_J \leq |x| \leq b_J$, $J = 1, 2, 3 ......n$.

[1]**Corresponding author**

If we assume that the upper and lower surfaces of the cracks are subjected to a prescribed pressure P(x) and temperature T(x), we see that inside the crack area, the following conditions are to be satisfied.

$$\sigma_{yy}(x, 0+) = \sigma_{yy}(x, 0-) = -P(x), \quad x \,\varepsilon\, U \tag{2.1}$$

$$\sigma_{xy}(x, 0+) = \sigma_{xy}(x, 0-) = 0, \quad x \,\varepsilon\, U \tag{2.2}$$

$$m_{\phi y}(x, 0+) = m_{\phi y}(x, 0-) = 0, \quad x \,\varepsilon\, U \tag{2.3}$$

$$T(x, 0+) = T(x, 0-) = -T(x), \quad x \,\varepsilon\, U \tag{2.4}$$

For the region of the interface not occupied by the cracks, the following continuity conditions must be satisfied:

$$u_x(x, 0+) = u_x(x, 0-), \quad x \,\varepsilon\, U' \tag{2.5}$$

$$u_y(x, 0+) = u_y(x, 0-), \quad x \,\varepsilon\, U' \tag{2.6}$$

$$\phi(x, 0+) = \phi(x, 0-), \quad x \,\varepsilon\, U' \tag{2.7}$$

$$\sigma_{xy}(x, 0+) = \sigma_{xy}(x, 0-), \quad x \,\varepsilon\, U' \tag{2.8}$$

$$\sigma_{yy}(x, 0+) = \sigma_{yy}(x, 0-), \quad x \,\varepsilon\, U' \tag{2.9}$$

$$m_{\phi y}(x, 0+) = \sigma_{\phi y}(x, 0-), \quad x \,\varepsilon\, U' \tag{2.10}$$

$$T(x, 0+) = T(x, 0-), \quad x \,\varepsilon\, U' \tag{2.11}$$

$$\overline{k_1}\frac{\partial T}{\partial Y}\bigg|_{y=0^+} = \overline{k_2}\frac{\partial T}{\partial Y}\bigg|_{y=0^-}, \quad x \,\varepsilon\, U' \tag{2.12}$$

Where U is the union of the cracks and U' its complement on the x-axis, i.e., $U = \bigcup_{J=1}^{n} (a_j, b_j)$, U' = R' – U, R' is the positive axis. The component of stress displacement and micro rotation must vanis as $\sqrt{x^2+y^2} \to 0$. cf. p. 29 (6) we take this condition as

$$u_y(x,0+) = 0\sqrt{(1-x)}, \quad x \,\varepsilon\, U'$$

In order to simplify the calculations, we choose P(x) to be an even function of x. Solution of the displacement equations can be taken by using Fourier sine and cosine transforms.

$$u_x(\xi,y) = \begin{cases} F_s\left[A_1 - P_1\xi^{-1}B_1 + Q_1B_1y\right]e^{-\xi y} - L_1^2\eta_1C_1e^{-\eta_1 y}, & y > 0 \\ F_s\left[A_2 - P_2\xi^{-1}B_2 + Q_2B_2y\right]e^{\xi y} - L_2^2\eta_2C_2e^{\eta_2 y}, & y < 0 \end{cases} \tag{2.13}$$

$$u_y(\xi,y) = \begin{cases} F_c[A_1 + Q_1B_1\,y]e^{-\xi y} - L_1^2\xi C_1e^{-\eta_1 y}, & y > 0 \\ F_c[A_2 + Q_2B_2y]e^{\xi y} + L_2^2\xi C_2e^{\eta_2 y}, & y < 0 \end{cases} \tag{2.14}$$

$$\phi(\xi,y) = \begin{cases} F_s[B_1x\,e^{-\xi y} + C_1e^{-\eta_1 y}], & y > 0 \\ F_s[B_2x\,e^{\xi y} + C_2e^{\eta_2 y}], & y < 0 \end{cases} \tag{2.15}$$

$$T_x(\xi,y) = \begin{cases} F_c\left[\dfrac{\Omega_1(\xi)}{\alpha_1(1+\eta_1)}e^{-\xi y}, \ \xi \to x\right], & y > 0 \\ F_c\left[\dfrac{\Omega_2(\xi)}{\alpha_2(1+\eta_2)}e^{\xi y}, \ \xi \to x\right], & y < 0 \end{cases} \tag{2.16}$$

Where

$$P_i = \frac{\lambda_i + 3\mu_i}{\lambda_i + 2\mu_i}, \ Q_i = \frac{\lambda_i + \mu_i}{\lambda_i + 2\mu_i}, L_i^2 = \frac{\nu_i}{2\mu_i}$$

$$\Gamma_1 = \frac{\mu_1}{\mu_2}, \ \Gamma_2 = \frac{\nu_1}{\nu_2}, \ P_i + Q_i = 2$$

Here λ_i and μ_i are the classical Lames' constants and ν_i is the micropolar modulii and Q_i is the micropolar poisson ration. The micropolar modulii ν_i and μ_i have the dimensions of force and stress respectively, we may define and internal characteristics length L_i of the medium given by

$$L_i = \sqrt{\frac{\nu_i}{2\mu_i}}$$

Now putting y = 0 in the equations (2.13 -2.15) we have

$$\sigma_{xy}(x, 0+) = F_s\left[2\mu_1\left(-\xi A_1 + B_1 + L_1^2\xi^2C_1\right); \ \xi \to x\right] \tag{2.17}$$

$$\sigma_{xy}(x, 0-) = F_s\left[2\mu_2\left(\xi A_2 + B_2 + L_2^2\xi^2C_2\right); \ \xi \to x\right]$$

$$\sigma_{yy}(x, 0+) = F_c\left[2\mu_1\{-\xi A_1 + (1-Q_1)B_1 + L_1^2\xi\eta_2C_2\}; \ \xi \to x\right] \tag{2.18}$$

σ_{yy} (x, 0-) =
$F_c\left[2\mu_2\{-\xi A_2+(1-Q_1)B_2+L_2^2\xi\eta_2 C_2\};\ \xi\to x\right]$

$m_{\phi y}(x, 0+) = F_s\left[-\nu_1\left(B_1\xi+C_1\eta_1\right);\ \xi\to x\right]$ (2.19)

TEMPERATURE FIELD

On applying the conditions (2.4),(2.11)and(2.12) we have

$$\phi_2(\xi)=\frac{\alpha_2(1+\eta_2)}{\alpha_1(1+\eta_1)}\phi_1(\xi) \quad (3.1)$$

and

$$F\left[\xi\left\{\frac{\overline{k_1}\phi_1(\xi)}{\alpha_1(1+\eta_1)}+\frac{\overline{k_2}\phi_2(\xi)}{\alpha_2(1+\eta_2)}\right\};\ \xi\to x\right]=0 \quad (3.2)$$

or

$$\xi F\left[\{\phi_1(\xi)\};\ \xi\to x\right]=0,\ \ldots\ldots,\ x\ \varepsilon\ U$$

Hence from condition (2.4) we get the following pair of dual integral equations.

$$F\left[\phi_1(\xi);\ \xi\to x\right]=-\alpha_1\left(1+\eta_1\right).T(x)$$

$$F\left[\phi_1(\xi);\ \xi\to x\right]=0 \quad (3.3)$$

Where F_c and F_s are Fourier cosine and sine transforms of F.

Now taking $\xi\phi_1(\xi)=\int_0^1\psi(t)\cos(\xi t)\,dt$

and it is easily shown that

$$\psi(t)=\frac{-2a_1(1+\eta_1)}{\pi}\frac{d}{dt}\int_0^t\frac{x.T(x)}{\sqrt{t^2-x^2}}dx \quad (3.4)$$

THERMO-ELASTIC PROBLEM

Now applying the conditions (2.1 to 2.3) and (2.8-2.10) in the equations (2.17-2.19) we get.

$$\xi A_2+(1-Q_2)B_2+L_2^2\xi C_2=\Gamma_1\left\{-\xi A_1+(1-Q)B_1+\eta_1 L_1^2\xi C_1\right\}$$

$$\xi A_2+B_2+L_2^2\xi^2 C_2=-\Gamma_1\left\{-\xi A_1+B_1+\xi^2 L_1^2 C_1\right\}$$

$$\xi B_2+\eta_2 C_2=\Gamma_2\left(\xi B_1+\eta_1 C_1\right)$$

Solving these equations for A_2, B_2, C_2 in terms of A_1, B_1, C_1 we find

$$A_2=-\Gamma_1\left[1+2\overline{Q^1}\{(1-Q_2)\eta_2+\eta_2\xi^2L_2^2\}\right]A_1$$

$$+\xi^{-1}\left[\Gamma_1(1-Q_1)-Q^{-1}(1-Q_2)\{\Gamma_2L_2^2\xi^2(\eta_2-\xi)-\Gamma_1\eta_2(2-Q_2)\}+\xi\eta_2L_2^2\{\Gamma_1\xi(2-Q_1)\Gamma_2\xi Q_2\}\right]B_1$$
$$+\left[\Gamma_1\eta_1L_1^2-Q^{-1}\left((1-Q_1)\{\Gamma_1\eta_1L_2^2(\eta_2-\xi)-\Gamma_2\eta_2L_1^2(\eta_1-\xi)\}+\eta_2L_2^2\{\Gamma_1\xi^2(\eta_1+\xi)L_1^2+\xi\Gamma_2\eta_1Q_2\}\right)\right]C_1$$

$$B_2=Q^{-1}\left[2\Gamma_1\eta_2\xi A_1+\left\{\Gamma_2L_2^2\xi^2(\eta_2-\xi)-\Gamma_1\eta_2(2-Q_1)\right\}B_1+\left\{\eta_1L_2^2\Gamma_2\xi(\eta_2-\xi)-\Gamma_1\eta_2L_1^2\xi(\eta_2+\xi)\right\}C_1\right]$$

$$C_2=Q^{-1}\left[-2\xi^2\Gamma_1A_1+\{\Gamma_2Q_2\xi+(2-Q_1)\xi\Gamma_1\}B_1+\left\{\Gamma_2Q_2\eta_1+\Gamma_1L_1^2\xi^2(\eta_1+\xi)C_1\right\}\right]$$

Where $Q=L_2^2\xi^2(\eta_2-\xi)+Q_2\eta_2$

Now using the condition of equations(2.5-2.7) we get

$$F_s\left[\xi(A_1+A_2)+\{(2-Q_2)B_1-(2-Q_1)B_1\}+L_2^2\xi\eta_2C_2-L_1^2\xi\eta_1C_1:x\right]=0 \quad (4.1)$$

$$F_c\left[A_1-A_2-\xi(L_1^2C_1+L_2^2C_2):x\right]=0 \quad (4.2)$$

$$F_s\left[B_1-B_2+C_1+C_2:x\right]=0,\quad x\ \varepsilon\ U' \quad (4.3)$$

Substituting the values of A_2, B_2, C_2 in the above equations (4.1-4.3) and then from boundary conditions (2.1-2.3) we get.

$$F_c\left[-\xi A_1+B_1+L_1^2\xi^2C_1:x\right]=\frac{p(x)}{2\mu_1},\quad x\ \varepsilon\ U$$

$$F_s\left[-\xi A_1+(1-Q_1)B_1+L_1^2\xi\eta_1C_1:x\right]=0,\quad x\ \varepsilon\ U \quad (4.4)$$

$$F_c\left[B_1\xi+\eta_1C_1:x\right]=0,\ x\ \varepsilon\ U$$

Solving the equations for A_1, B_1, C_1 we obtain

$$aA_1=\{c_1\phi(\xi)-C_1\psi(\xi)\}(b_2c_3-b_3c_2)-\{c_3\psi(\xi)-c_2X(\xi)\}(b_1c_2-b_2c_1)$$

$aB_1 = (a_1c_2 - a_2c_1)\{c_3\psi(\xi) - c_2X(\xi)\} - (a_2c_3 - a_3c_2)\{c_2Q(\xi) - c_3\psi(\xi)\}$

$ac_1C_1 = a\phi(\xi) - (a_1D_1 + b_1D_2)$

Where

$a = (a_1c_2 - a_2c_1)(b_2c_3 - b_3c_2) - (a_2c_3 - a_3c_2)(b_1c_2 - b_2c_1)$

$$a_1 = 1 - \Gamma_1 + 2\Gamma_1Q^{-1}(\eta_2(2 - Q_2) - \xi^2L_2^2\eta_2 - \eta_2\{(1 - Q_2) + L_2^2\xi^2\})$$

$$\begin{aligned} b_1 = {} & L_2^2\eta_2Q^{-1}\xi\{\Gamma_2Q_2 + \Gamma_1(2 - Q_1)\} + \xi^{-1}\Gamma_1(1 - Q_1) \\ & -Q^{-1}\big[(1 - Q_2)\{\Gamma_2L_2^2\xi^2(\eta_2 - \xi) - \Gamma_1\eta_2(2 - Q_2)\} \\ & + \xi^2L_2^2\eta_2\{\Gamma_1(2 - Q_1) + \Gamma_2Q_2\}\big] \\ & + (2 - Q_2)\xi^{-1}Q^{-1}\{\Gamma_2L_2^2\xi^2(\eta_2 - \xi) \\ & - \Gamma_1\eta_2(2 - Q_1)\} - \xi^{-1}(2 - Q_1) \end{aligned}$$

$$\begin{aligned} c_1 = {} & \Gamma_1\eta_1L_1^2 - Q^{-1}\big[(1 - Q_2)\{\eta_1\Gamma_1L_2^2(\eta_2 - \xi) \\ & - \Gamma_2\eta_2L_1^2(\eta_1 - \xi)\} \\ & + \eta_2L_2^2\{\Gamma_1\xi^2(\eta_1 + \xi)L_1^2 + \Gamma_2\eta_1Q_2\}\big] \\ & + (2 - Q_2)\xi^{-1}Q^{-1}\{\Gamma_2\eta_1L_1^2(\eta_1 + \xi)\} \\ & (\eta_2 + \xi) - \Gamma_1\eta_1L_1^2(\eta_1 + \xi)) \\ & + L_2^2\eta_2Q^{-1}\{\Gamma_2Q_2\eta_1 + \Gamma_1L_1^2\xi^2(\eta_1 + \xi) \\ & - L_1^2\eta_1\} \end{aligned}$$

$$a_2 = 1 + \Gamma_1 + 2Q^{-1}\big[\Gamma_1\eta_2(1 - Q_2) + \Gamma_1\eta_2\xi^2L_2^2 + \xi^3L_2^2\big]$$

$$\begin{aligned} b_2 = {} & Q^{-1}\big[(1 - Q_2)\{\Gamma_2\Gamma_2^2\xi(\eta_2 - \xi) - \Gamma_1\xi^{-1}\eta_2(2 - Q_1)\} \\ & + L_2^2\eta_2\{\Gamma_1\xi(2 - Q_1) + \Gamma_2\xi Q_2\} \\ & + L_2^2\xi^2\{\Gamma_2Q_2 + (2 - Q_1)\Gamma_1\}\big] \\ & - \xi^{-1}\Gamma_1(1 - Q_1) \end{aligned}$$

$$\begin{aligned} c_2 = {} & Q^{-1}\big[(1 - Q_2)\{\Gamma_1\eta_1L_2^2(\eta_2 - \xi) - \Gamma_2\eta_2L_1^2(\eta_1 + \xi) \\ & + \eta_2L_2^2\}\{\Gamma_1\xi^2(\eta_1 + \xi)L_1^2 + \Gamma_2\eta_1Q_2\} \\ & + L_2^2\xi\{\Gamma_2Q_2\eta_1 + \Gamma_1L_1^2\xi^2(\eta_1 + \xi)\} + L_1^2\xi \\ & - \Gamma_1\eta_1L_1^2\big] \end{aligned}$$

$$a_3 = -2\Gamma_1\xi Q^{-1}(\eta_2 + \xi)$$

$$\begin{aligned} b_3 = {} & 1 + Q^{-1}\big[\Gamma_2Q_2\xi + (2 - Q_1)\xi\Gamma_1 \\ & - \{\Gamma_2\Gamma_2^2\xi^2(\eta_2 - \xi) - \Gamma_1\eta_2(2 - Q_1)\}\big] \end{aligned}$$

$$\begin{aligned} c_3 = {} & 1 + Q^{-1}\Big[\Gamma_2Q_2\eta_1 + \Gamma_1L_1^2\xi^2(\eta_2 + \xi) \\ & - \xi\Big\{\Gamma_2\eta_1\Gamma_2^2(\eta_2 - \xi) - \Gamma_1\eta_2(\eta_1 + \xi)\Big\}\Big] \end{aligned}$$

$D_1 = (b_2c_3 - b_3c_2)\{c_2\phi(\xi) - c_1\psi(\xi)\} - (b_1c_2 - b_2c_1)\{c_3\psi(\xi) - c_2X(\xi)\}$

$D_2 = (a_1c_2 - a_2c_1)\{c_3\psi(\xi) - c_2X(\xi)\} - (a_2c_3 - a_3c_2)\{c_2\phi(\xi) - c_1\psi(\xi)\}$

Putting the values of A_1, B_1, C_1in the equation (4.4) which are reduced to the following sets of equations:

$F_c(a(\xi)\phi(\xi) + b(\xi)\psi(\xi) + c(\xi)X(\xi) : x) = f(x)$

$F_s(b(\xi)\phi(\xi) + c(\xi)\psi(\xi) + a(\xi)X(\xi) : x) = 0$ (4.5)

$F_c(c(\xi)\phi(\xi) + a(\xi)\psi(\xi) + b(\xi)X(\xi) : x) = 0$

where,

$a(\xi) = a^{-1}(a_2c_3 - a_3c_2)c_2 - \xi a^{-1}(b_2c_3 - b_3c_2)c_2 + L_1^2\xi^2c_1^{-1}$

$b(\xi) = a^{-1}\{(b_2c_3 - b_3c_2)c_1 + (b_1c_2 - b_2c_1)c_3\}$
$\quad + a^{-1}\{(a_1c_2 - a_2c_1)c_3 + (a_2c_3 - a_3c_2)c_1\}$

$c(\xi) = a^{-1}(a_2c_1 - a_1c_2)c_2 - \xi a^{-1}\{(b_1c_2 - b_2c_1)\}c_2$

$f(x) = \frac{ap(x)}{2\mu_1}$

$\phi(\xi) = a_1A_1 + b_1B_1 + c_1C_1$

$\psi(\xi) = a_2A_1 + b_2B_1 + c_2C_1$

$X(\xi) = a_3 A_1 + b_3 B_1 + c_3 C_1$

And

$F_s(\phi(\xi) : x) = 0$

$F_s(\psi(\xi) : x) = 0$
(4.7)

$F_s(X(\xi) : x) = 0$

If we differentiate (4.7) with respect to x we see that $\phi(\xi)$, $\psi(\xi)$ and $X(\xi)$ must satisfy the equivalence relations (cf. 734 of (41))

$F_c(a(\xi)\phi(\xi) + b(\xi)\Psi(\xi) + c(\xi)X(\xi) : x) = f(x)$

$F_s(b(\xi)\phi(\xi) + c(\xi)\Psi(\xi) + a(\xi)X(\xi) : x) = 0, x \varepsilon U'$

$F_c(c(\xi)\phi(\xi) + a(\xi)\Psi(\xi) + b(\xi)X(\xi) : x) = 0$
(4.8)

$F_c(\phi(\xi) : x) = 0$ (4.6)

$F_s(\Psi(\xi) : x) = 0$, $x \varepsilon U'$

$F_c(X(\xi) : x) = 0$

The set of simultaneous integral equations (4.8) can be reduced to Riemann boundary value problem.

If we suppose

$$F_c(\phi(\xi) : x) = \begin{cases} r_1(x), & x \varepsilon U \\ 0, & x \varepsilon U' \end{cases}$$

$$F_s(\Psi(\xi) : x) = \begin{cases} s_1(x), & x \varepsilon U \\ 0, & x \varepsilon U' \end{cases}$$

$$F_c(X(\xi) : x) = \begin{cases} w_1(x), & x \varepsilon U \\ 0, & x \varepsilon U' \end{cases} \qquad (4.9)$$

Here F_c and F_s represent Fourier cosine and sine transforms and $U = \cup_{j=1}^{n}(a_j, b_J)$, $U' = R' - U$, R' is the positive real axis, $0 \le a_1 < b_1 < a_2 < b_2 \ldots\ldots < a_n < b_n$

Now using (5), it can be easily shown that

$$F_s(\phi(\xi) : x) = \frac{1}{\pi}\int_L \frac{r(u)}{x-u}\,du$$

$$F_c(\Psi(\xi) : x) = -\frac{1}{\pi}\int_L \frac{s(u)}{x-u}\,du \qquad (4.10)$$

$$F_s(X(\xi) : x) = \frac{1}{\pi}\int_L \frac{w(u)}{x-u}\,du$$

where r(u), s(u) and w(u) are odd and even extensions of $r_1(u)$, $s_1(u)$ and $w_1(u)$ respectively to the interval U. Using (4.9) and (4.10) the set of integral equations (4.8) reduced to simultaneous singular integral equations.

$$a(\xi) - r(x)\frac{b(\xi)}{\pi}\int_L \frac{s(u)}{u-x}\,du + c(\xi)w(x) = f(x)$$

$$a(\xi) - s(x)\frac{b(\xi)}{\pi}\int_L \frac{w(u)}{u-x}\,du + c(\xi)r(x) = 0 \qquad (4.11)$$

$$a(\xi) - w(x)\frac{b(\xi)}{\pi}\int_L \frac{r(u)}{u-x}\,du + c(\xi)\,s(x) = 0$$

If we write $\lambda(x) = s(x) - ir(x) + w(x)$ (4.12)

Then (4.11) reduce to the singular integral equation

$$i\,a(\xi)\,\lambda(x) + \frac{c(\xi) - b(\xi)}{\pi}\int_L \frac{\lambda(u)}{u-x}\,du = f(x),\ x \varepsilon U \qquad (4.13)$$

where $f(x) = \widehat{f_1}(x) + \widehat{f_2}(x)$.

If we define

$$\Lambda(z) = \frac{1}{2\pi i}\int_U \frac{\lambda(u)}{u-z}\,du$$

and on using the Sokhotski formulae (8)

$\Lambda^{+} - \Lambda^{-} = \lambda(x),\ \Lambda^{+} - \Lambda^{-} = \frac{1}{2\pi_i}\int_U \frac{\lambda(u)}{u-x}\,du$ (4.14)

then(4.13) reduces to the Riemann boundary value problem)

$\Lambda^{+}(x) + k\Lambda^{-}(x) = i\{c(\xi) - b(\xi) + a(\xi)\}^{-1} f(x) x \varepsilon U$ (4.15)

where $k = \left(\frac{c(\xi) - b(\xi) - a(\xi)}{c(\xi) - b(\xi) + a(\xi)}\right) > 0$ (4.16)

The solution of this problem is well known (13) and is given by

$\Lambda(z) = \frac{X(z)}{2\pi}\{c(\xi) - b(\xi) + a(\xi)\}^{-1}\left[\int_U \frac{f(t)}{x^{+}(t)(t-z)}\,dt + p(z)X(z)\right]$ (4.17)

where

$p(z) = h_1 z^{n-1} + h_2 z^{n-2} + ... + h_n$ (4.18)

$h_1, h_2, h_3,$ ....h_n are arbitrary complex constants and X(z) is the solution of homogeneous Riemann problem-

$X^{+}(t) + K X^{-}(t) = 0, t \ \varepsilon\ U$ (4.19)

This problem has known solution (13)

$Xz(\,) = \prod_{J=1}^{n}\left((z-a_J)(z-b_J)\right)^{-½-i\omega}\left[(z+a_J)(z-b_J)\right]^{-½-i\omega}, a_1 \neq 0$

$= (z-b_J)^{-½-i\omega}(z-b_1)^{-½+i\omega}\prod_{J=1}^{n}\left((z-a_J)(z+b_J)\right)^{-½-i\omega}$

$((z+a_J)(z-b_J))^{-½-i\omega}\ a_1 = 0$ (4.20)

where$\omega = \frac{1}{2p}\log\left(\frac{c(\xi) - b(\xi) - a(\xi)}{c(\xi) - b(\xi) + a(\xi)}\right)$

In case f(x) is a polynomial (5, p, 1030)

$\int_U \frac{f(t)}{x^{+}(t)(t-z)}\,dt = \pi i\left(\frac{c(\xi) - b(\xi)}{c(\xi) - b(\xi) + a(\xi)}\right)\left[\frac{f(z)}{X(z)} - L(z)\right]$ (4.21)

where

$L(z) = \frac{1}{2\pi}\lim_{R\to\infty}\int_0^{2\pi}\frac{f(Re^{i\theta}).Re^{i\theta}}{X(Re^{i\theta}).(Re^{i\theta}-z)}\,d\theta$ (4.22)

Hence (4.17) yield

$\Lambda(z) = \frac{i}{2b(\xi)}(f(z) - L(z)) + P(z)X(z)$ (4.23)

Using (4.9) and 4.10) it can be shown that-

$\sigma_{yy}(x, 0+) = \sigma_{yy}(x, 0-) = f_0 b(\xi)\mu_1\int_U \frac{s(u)}{u-x}\,du$

$\sigma_{xy}(x, 0+) = \sigma_{xy}(x, 0-) = f_0 b(\xi)\mu_1\int_U \frac{r(u)}{u-x}\,du, x\ \varepsilon\ U$

$m_{\phi y}(x, 0+) = m_{\phi y}(x, 0-) = f_0 b(\xi)\mu_1\int_U \frac{w(u)}{u-x}\,du$

As given in (7) the continuity conditions(2.5) are satisfied if

$F_s(\phi(\xi) : x) = 0$

$F_c(\Psi(\xi) : x) = 0, \quad x\ \varepsilon\ U'$

$F_s(X(\xi) : x) = 0$

Substituting the values of $\phi(\xi)$, $\Psi(\xi)$ and $X(\xi)$ from (4.9) and interchanging the order of integrations we obtain

$\int_{a_J}^{b_J} s_1(u)\,du = 0$

$\int_{a_J}^{b_J} r_1(u)\,du = 0, J = 1,2,3........n$ (4.25)

$\int_{a_J}^{b_J} w_1(u)\,du = 0$

Using (4.12) we obtain

$\int_{a_J}^{b_J} \lambda_1(u)\,du = 0, J = 1,2,3...........n$ (4.26)

CONSTANT TEMPERATURE

If we consider the physical important case in which the cracks are opened by the application of a prescribed constant temperature T_0 at it surfaces then,

$T(x) = T_0(x)$ = constant and from the equation (3.4) we have

$$\Psi(t) = \frac{-2\alpha_1(1+\eta_1)}{\pi} = T_1 \text{ (say)}$$

Hence, $\phi_1(\xi) = \frac{T_1 \sin\xi}{\xi^2}$ (5.1)

We shall also assume that there is no external pressure applied to the surfaces of the cracks, so that p(x) = 0. Thus we have

$$\widehat{f_1}(x) = e_1\sqrt{\frac{2}{\pi}}\int_0^\infty \phi_1(\xi)\sin(\xi x)\cos(\xi t)\,d\xi dt = e_1\sqrt{\frac{2}{\pi}}\,.\,x \quad (5.2)$$

$$\widehat{f_2}(x) = e_2\sqrt{\frac{\pi}{2}}\left[\log[1-x^2] + x\log\left|\frac{1+x}{1-x}\right|\right] + C_1$$

Where

$$e_1 = -\frac{2k_1\Gamma((\Gamma+k_2)\alpha_1(1+\eta_1)+(1+k_1)\alpha_2(1+\eta_1))}{\alpha_1(1+\eta_1)} \quad (5.3)$$

$$e_2 = -\frac{2k_1\Gamma((\Gamma+k_2)\alpha_1(1+\eta_1)+(1+k_1)\alpha_2(1+\eta_1))}{\alpha_1(1+\eta_1)}$$

Thus $g(x) = e_1T_1x\left(\frac{\pi}{2}\right)^{1/2} - \frac{iT_1e_2}{(2\pi)^{1/2}}\left(x^2+\frac{x^4}{6}\right) + C$ (5.4)

Now substituting the value of g(t) and x(t) in (4.21) and (4.23) we have

$$s(x) = -\{b(\xi)^2 - a(\xi)^2\}^{-1/2}[(d_1x^3+d_2x+d_3)\cos\omega\theta + (cx^4+c'x^2+c'')\sin\omega\theta] \quad (5.5)$$

$$r(x) = -\{b(\xi)^2 - a(\xi)^2\}[(d_1x^3+d_2x+d_3)\sin\omega\theta + (cx^4+c'x^2+c'')\cos\omega\theta] \quad (5.6)$$

$$w(x) = \{b(\xi)^2 - a(\xi)^2\}[(d_1x^3+d_2x+d_3)\cos\omega\theta + (cx^4+c'x^2+c'')\sin\omega\theta] \quad (5.7)$$

Where d_1, d_2, d_3 are no constants while c' and c" are unknown to be determined. Since, s(x), r(x) and w(x) are odd and even, we have c' = O. On using (4.25) we have

$$c'' = \frac{\int_a^b [(d_1x^3+d_2x)\sin\omega\theta + (cx^4+cx^2)\cos\omega\theta]\,dx}{\int_a^b \cos\omega\theta\, d\theta} \quad (5.8)$$

PARTICULAR CASES

Case-I: Single Crack at the interface opened by constant pressure

In the case of single Crack L_1 is to be taken the interval (0, 1). From 4.8) we get simultaneous dual) integral equations.

$F_c(a(\xi)\phi(\xi) + b(\xi)\Psi(\xi) + c(\xi)X(\xi) : x) = f(x)$

$F_s(b(\xi)\phi(\xi) + c(\xi)\Psi(\xi) + a(\xi)X(\xi) : x) = 0,\ 0 \le x \le 1$

$F_c(c(\xi)\phi(\xi) + \Psi(\xi) + b(\xi)X(\xi) : x) = 0$ (6.1)

$F_s(\Psi(\xi) : x)$

$F_c(\phi(\xi) : x),\ x > 1,$ (6.2)

$F_s(X(\xi) : x)$

Here the crack is defined by y = 0, -1 ≤ x ≤ 1. Now taking n = 1 we obtain from (4.20), (4.18),(4.22), (4.23)

$X(z) = (z+1)^{-1/2+i\omega}(z+1)^{-1/2-i\omega}p(z) = h_1$ (6.3)

$L(z) = (z-2i\omega)f_0$

$\Lambda(z) = if_0\{2b(\xi)\}^{-1}(1-(z+h_1)X(z))$

where f_0 is a constant such that $f(x) = f_0$ is given by

$f_0 = ((a_1c_2 - a_2c_1)(b_2c_3 - b_3c_2) - (a_2c_3 - a_3c_2)(b_1c_2 - b_2c_1))\ \frac{p_0}{2\mu_1}$

And it is easily shown that

$$r(x) = -\frac{f_0}{2b(\xi)}\left[\frac{c(\xi)-b(\xi)+a(\xi)}{c(\xi)-b(\xi)-a(\xi)}\right]^{1/2}(1-x^2)^{-1/2}\{x\sin\omega\theta - \omega\cos\omega\theta\}$$

$$s(x) = -\frac{f_0}{2b(\xi)}\left[\frac{c(\xi)-b(\xi)+a(\xi)}{c(\xi)-b(\xi)-a(\xi)}\right]^{1/2}(1-x^2)^{-1/2}\{x\cos\omega\theta - \omega\sin\omega\theta\}$$

$$w(x) = -\frac{f_0}{2b(\xi)}\left[\frac{c(\xi)-b(\xi)+a(\xi)}{c(\xi)-b(\xi)-a(\xi)}\right]^{1/2}(1-x^2)^{-1/2}\{x\cos\omega\theta - \sin\omega\theta\} \quad (6.4)$$

$$\sigma_{yy}(x\,0+) = -p_0(1-(1-x^2)^{-1/2}(x\cos\omega\theta - \sin\omega\theta)$$

$$\sigma_{xy}(x\,0+) = p_0((1-x^2)^{-1/2}(\cos\omega\theta + x\sin\omega\theta) \quad (6.5)$$

$$m_{\phi y}(x\,0+) = -p_0((1-x^2)^{-1/2}(\omega\cos\omega\theta + x\sin\omega\theta)$$

Case-II Two Collinear Griffith Cracks

In the case of two collinear Griffith cracks, we choose $L_1 = (a, b)$ and the cracks are given by $y = 0$, $a \leq x \leq b$, $a \neq 0$. In this case, the equations (4.8) reduce to the following set of simultaneous triple integral equations.

$$F_c(\phi(\xi) : x) = 0$$
$$F_s(\Psi(\xi), x) = 0 \qquad 0 < x < a \qquad (6.6)$$
$$F_c(X(\xi), x) = 0$$
$$F_c(a(\xi)\phi(\xi) + b(\xi)\Psi(\xi) + c(\xi)X(\xi) : x) = f(x)$$
$$F_s(b(\xi)\phi(\xi) + c(\xi)\Psi(\xi) + a(\xi)X(\xi) : x) = a, \; 0 \leq x \leq b$$
$$F_c(c(\xi)\phi(\xi) + a(\xi)\Psi(\xi) + b(\xi)X(\xi)) = 0 \qquad (6.7)$$
$$F_c(\phi(\xi), x) = 0$$
$$F_s(\Psi(\xi), x) = 0, \quad x > b \qquad (6.8)$$
$$F_c(X(\xi), x) = 0$$

In this case we may write the following from (4.18), (4.20), (4.22) ,(4.23) on putting n = 1.

$$x(z) = ((z - a)((z - b)))^{-½+i\omega} - ((z + a)((z - b)))^{-½-i\omega} - P(z) = h_1 z + h_2 \qquad (6.9)$$

and L(z) is the constant term in the expansion of $\frac{t f_0}{X(t)(t-z)}$.

Now, $$\frac{t f_0}{X(t)(t-z)}$$
$$= f_0 \frac{1}{(1-z/t)}\{(t^2-b^2)(t^2 - a^2)\}^{½}\left[\frac{(t-b)(t+a)}{(t+b)(t-a)}\right]^{i\omega}$$
$$= t^2 f_0\left(1-\frac{z}{t}\right)^{-1}\left(1-\frac{b^2}{t^2}\right)^{½}\left(1-\frac{a^2}{t^2}\right)^{½}$$
$$.\left(1-\frac{b}{t}\right)^{i\omega}\left(1+\frac{a}{t}\right)^{i\omega}\left(1+\frac{b}{t}\right)^{i\omega}\left(1-\frac{a}{t}\right)^{-i\omega}$$
$$= t_0 f_0\left[1+\frac{z}{t}-\frac{z^2-a^2-b^2}{2t^2}\right]\left[1+2i\omega(a-b)-\frac{2\omega^2}{t^2}(a-b)^2\right] \qquad (6.10)$$

Therefore constant term in this expansion is

$$L(z) = f_0(z^2 + g_1 z + g_2) \qquad (6.11)$$

Where

$$g_1 = 2i\omega(a-b),\; g_2 = 2\omega^2(a-b)^2 - \frac{a^2+b^2}{2}$$

are the constants and hence

$$\Lambda(z) = \frac{i f_0}{\{\omega\, b(\xi)\}}[1 - \{z^2 + p(z)X(z)\}Xz] \qquad (6.12)$$

where h_1 and h_2 are arbitrary complex constants. Now for $a \leq |x| \leq b$, we have

$$X^+(x) = i\sqrt{\frac{k}{(b^2-x^2)(x^2-a^2)}}\,(\cos\omega\theta + i\sin\omega\theta)$$

$$X^-(x) = \frac{i}{\sqrt{K(b^2-x^2)(x^2-a^2)}}\,(\cos\omega\theta + i\sin\omega\theta)$$
(6.13)

where, $\theta = \log\left(\frac{(x-a)(x+b)}{(x+a)(b-x)}\right)$

Hence

$$\Lambda^+(x) - \Lambda^-(x) = \frac{i f_0}{2b(\xi)}(x_2 + h_1 x + h_2)(X^- - X^+)$$
$$= -f_0[\{b(\xi)^2 - a(\xi)^2\}\{(b^2-x^2)(x^2-a^2)\}]^{-½}$$
$$.i(x^2 + h_1 x + h_2)(\cos\omega\theta + i\sin\omega\theta)$$
(6.14)

and

$$\Lambda^+(x) + \Lambda^-(x) = \frac{i f_0}{2b(\xi)}(2-(x^2 - h_1 x + h_2)(X^+ - X^-)$$
$$= i f_0 b(\xi)^{-1}\{1 - (b^2-x^2)(x^2-a^2)\}^{-½}$$
$$.i(x^2 + h_1 x + h_2)(\cos\omega\theta + i\sin\omega\theta)$$
(6.15)

Now from (4.14)

$$S(x) = -f_0[\{b(\xi)^2 - a(\xi)^2\}\{(b^2-x^2)(x^2-a^2)\}^{-½}(x^2 + h_1' x + h'_2)]\cos\omega\theta - (h''_1 + h''_2)\sin\omega\theta)$$

$$r(x) = -f_0[\{b(\xi)^2 - a(\xi)^2\}\{(b^2-x^2)(x^2-a^2)\}^{-½}(x^2 + h_1' x + h'_2)]\sin\omega\theta - (h''_1 + h''_2)\cos\omega\theta),\; a < |x| < b$$

(6.16)

$$w(x) = f_0[\{b(\xi)^2 - a(\xi)^2\}\{(x^2-a^2)(b^2-x^2)\}^{-½}(x^2 + h_1' x + h'_2)]\sin\omega\theta - (h''_1 + h''_2)\cos\omega\theta)$$

where

$$h_1 = h'_1 + i\,h''_1$$
$$h_2 = h'_2 + i\,h''_2$$

$$\frac{1}{\pi i}\int_{L_1}\frac{\lambda(u)}{(u-x)}\,du = \Lambda^+(x) + \Lambda^-(x) \tag{6.17}$$

$$\frac{1}{\pi i}\int_{L_1}\frac{r(u) - is(u)}{(u-x)}\,du = i f_0\{b(\xi)^{-1}\}(1 - i\{(b^2-x^2)(x^2-a^2)\}^{-1/2} . \{(x^2+h_1x+h_2)(\cos\omega\theta + i\sin\omega\theta) \tag{6.18}$$

Separating into real and imaginary parts, we have

$$\frac{1}{\pi}\int_{L_1}\frac{r(u)}{(u-x)}\,du = -f_0\{b(\xi)^{-1}\}1 + a\{\xi\}\{(b^2-x^2)(x^2-a^2)\}^{-1/2} \{(x^2+h'_1x+h'_2)\sin\omega\theta + (h''_1x+h''_2)\cos\omega\theta\}) \tag{6.19}$$

$$\frac{1}{\pi}\int_{L_1}\frac{r(u)}{(u-x)}\,du = f_0\frac{a(\xi)}{b(\xi)}\{(b^2-x^2)(x^2-a^2)\}^{-1/2}.\{(x^2+h'_1x+h'_2)\cos\omega\theta + (h''_1x+h''_2)\sin\omega\theta)$$

$$\frac{1}{\pi}\int_{L_1}\frac{w(u)}{(u-x)}\,du = f_0\{b(\xi)^{-1}\}[1-a\{\xi\}]\{(b^2-x^2)(x^2-a^2)\}^{-1/2}\}\{(x^2+h'_1x+h'_2)\sin\omega\theta + (h''_1x+h''_2)\cos\omega\theta)$$

Again for $0 \le |x| \le a$, $b \le |x| \le a$ i.e. $x\,\varepsilon L'_1$

$$\frac{1}{2\pi i}\int_{L_1}\frac{\lambda(u)}{u-x}\,du = \Lambda^+(x) - \Lambda^-(x) \tag{6.20}$$

Therefore,

$$\frac{1}{2\pi i}\int_{L_1}\frac{s(u) - ir(u)}{u-x}\,du = \frac{2if_0}{b(\xi)}\left(1-(x^2+h_1x+h_2)X^+(x)\right)$$

$$= \frac{2if_0}{b(\xi)}\left((1-\{(x^2-b^2)(x^2-a^2)\}^{-1/2}(x^2+h_1x+h_2)\right) (\cos\omega\theta + i\sin\omega\theta) \tag{6.21}$$

Separating real and imaginary parts, we get

$$\frac{1}{\pi}\int_{L_1}\frac{s(u)}{u-x}\,du = -\frac{f_0}{b(\xi)}\left(1-\{(x^2-a^2)(x^2-b^2)\}^{-1/2}\right. \{(x^2+h'_1x+h'_2)\cos\omega\theta - (h''_1x+h''_2)\sin\omega\theta\})$$

$$\frac{1}{\pi}\int_{L_1}\frac{s(u)}{u-x}\,du = +\frac{f_0}{b(\xi)}\left(1+\{(x^2-a^2)(x^2-b^2)\}^{-1/2}\right. \{(x^2+h'_1x+h'_2)\sin\omega\theta - (h''_1x+h''_2)\cos\omega\theta\}) \tag{6.22}$$

$$\frac{1}{\pi}\int_{L_1}\frac{w(u)}{u-x}\,du = +\frac{f_0}{b(\xi)}\left(1+\{(x^2-a^2)(x^2-b^2)\}^{-1/2}\right. \{(x^2+h'_1x+h'_2)\cos\omega\theta - (h''_1x+h''_2)\sin\omega\theta\})$$

From (4.24) the stresses are given by

$$\sigma_{yy}(x, 0+) = -2b(\xi)\mu_1[\{(a_1c_2-a_2c_1)(b_2c_3-b_3c_2) - (a_2c_3-a_3c_2)(b_1c_2-b_2c_1)\}) \; F_c(\Psi(\xi):x)$$

$$= -2b(\xi)\mu_1[\{(a_1c_2-a_2c_1)(b_2c_3-b_3c_2) - (a_2c_3-a_3c_2)(b_1c_2-b_2c_1)\})^{-1} \frac{1}{\pi}\int_{L_1}\frac{s(u)}{u-x}\,du$$

$$= P_0[1-\{(x^2-b^2)(x^2-a^2)\}^{-1/2}\{(x^2+h'_1x+h'_2)\cos\omega\theta - (xh''_1+h''_2)\sin\omega\theta\}] \tag{6.23}$$

$$\sigma_{xy}(x, 0+) = -2\mu_1 b(\xi)[\{(a_1c_2-a_2c_1)(b_2c_3-b_3c_2) - (a_2c_3-a_3c_2)(b_1c_2-b_2c_1)\}) \; F_s(\Psi(\xi):x)$$

$$= -2\mu_1 b(\xi)[\{(a_1c_2-a_2c_1)(b_2c_3-b_3c_1) - (a_2c_3-a_3c_2)(b_1c_2-b_2c_1)\})^{-1} \frac{1}{\pi}\int_{L_1}\frac{r(u)}{u-x}\,du$$

$$= P_0[1-\{(x^2-b^2)(x^2-a^2)\}^{-1/2}\{(x^2+h'_1x+h'_2)\sin\omega\theta - (xh''_1+h''_2)\cos\omega\theta\}] \tag{6.24}$$

$$m_{\phi y}(x, 0+) = 2\mu_1 b(\xi)[\{(a_1c_2-a_2c_1)(b_2c_3-b_3c_2) - (a_2c_3-a_3c_2)(b_1c_2-b_2c_1)\})^{-1} \; F_c(\Psi(\xi):x)$$

$$= 2\mu_1 b(\xi)[\{(a_1c_2-a_2c_1)(b_2c_3-b_3c_2) - (a_2c_3-a_3c_2)(b_1c_2-b_2c_1)\})^{-1}$$

$\frac{1}{\pi}\int_{L_1} \frac{w(u)}{u-x}\, du$

$= -P_0[1 + \{(x^2 - b^2)(x^2 - a^2)\}^{-½}\ \{(x^2 + h'_1 x + h'_2)\cos\omega\theta - (h''_1 + h''_2)\sin\omega\theta\})]$ (6.25)

For large x, we have

$\sigma_{yy}(x, 0+) = P_0\left[1 - \frac{1}{x^2}\left(1 - \frac{a^2}{x^2}\right)^{-½}\left(1 - \frac{b^2}{x^2}\right)^{-½}\left\{(x^2 + h'_1 x + h'_2)\left(1 - \frac{\omega^2\theta^2}{2} + \frac{\omega^4\theta^4}{4}\right) - (h''_1 + h''_2)\left(w\theta - \frac{\omega^3\theta^3}{6}\right)\right\}\right]$

$= P_0\left[1 - \left(\frac{1}{x^2} + \frac{a^2b^2}{2x^4}\right)\{x^2 + h'_1 x + h'_2 - 2\omega(b-a)h''_1 - 2\omega^2(b - a^2) + 0(x^{-1})\right]$

$= P_0\left[1 - \left\{1 - \frac{h_1}{x} + 0(x^{-1})\right\}\right]$

$= -P_0\left[\frac{h_1}{x} + 0(x^{-2})\right]$

(6.26)

Similarly,

$\sigma_{xy}(x, 0+) = P_0\left[\frac{1}{x}(h''_1 + 2\omega(b-a) + 0\,(x^{-2})\right]$ (6.26)

$m_{\phi y}(x, 0+) = P_0\left[\frac{h_1}{x} - 0(x^{-2})\right]$ (6.27)

Therefore, we have

$h_1 = 0$ and $h'_1 + 2\omega(b-a) = 0$

But r(x), s(x) and w(x) are even and odd functions of x, we get $h''_2 = 0$ and hence

$s(x) = -f_0[\{b(\xi)^2 - a(\xi)^2\}\{(b^2 - x^2)(x^2 - a^2)\}^{-½}\{(x^2 + h'_2)\cos\omega\theta + 2\omega x\,(b-a)\sin\omega\theta]$

$r(x) = -f_0[\{b(\xi)^2 - a(\xi)^2\}\{(b^2 - x^2)(x^2 - a^2)\}^{-½}\{(x^2 + h'_2) + \sin\omega\theta + 2\omega x\,(b-a)\sin\omega\theta]$

$w(x) = -f_0\,(\{b(\xi)^2 - a(\xi)^2\}\{(b^2 - x^2)(x^2 - a^2)\}^{-½}\{(x^2 + h'_2)\sin\omega\theta$

(6.28)

and

$\lambda(x) = s(x) - i\,r(x) = -f_0\,(\{b(\xi)^2 - a(\xi)^2\}\{(b^2 - x^2)(x^2 - a^2)\}^{-½}\ \{x^2 - 2i\,\omega(b-a)x + h'_2\}e^{i\omega\theta}$ (6.29)

The constant h'_2 can be calculated from the condition (6.12) which in this case is $\int_a^b s(t)\,dt = 0$ from where we get.

$\int_a^b \{(b^2 - x^2)(x^2 - a^2)\}^{-½}\{(x^2 + h'_2)\cos\omega\theta + 2w\,(b-a)\,x\sin\omega\theta\}dx = 0$

(6.30)

This gives

$h'_2 = \{2\omega\,(a - b)I_1 - I_2\}/I_0$ (6.31)

Where

$I_0 = \int_a^b \frac{\cos\omega\theta}{\sqrt{(x^2-a^2)(b^2-a^2)}}dx$

$I_1 = \int_a^b \frac{x\sin\omega\theta}{\sqrt{(b^2-x^2)(x^2-a^2)}}dx$

$I_2 = \int_a^b \frac{x^2\sin\omega\theta}{\sqrt{(b^2-x^2)(x^2-a^2)}}dx$ (6.32)

To calculate the above integrals we put $x = a\cos^2\theta + b\sin^2\theta$ and separating real and imaginary parts by using binomial expansion in the following integrals.

$\int_a^b \frac{e^{i\omega\theta}}{\sqrt{(x^2-a^2)(b^2-x^2)}}dx = \int_a^b (x-a)^{-½+i\omega}(x+a)^{-½+i\omega}(b+x)^{-½-i\omega}$

$(b-x)^{-½-i\omega}dx$

$= \frac{\pi}{(a+b)\cosh\pi\omega}F_3(½ + i\omega, ½ - i\omega, ½i\omega, 1, z, -z)$ (6.33)

$\int_a^b \frac{x\,e^{i\omega\theta}}{\sqrt{(x^2-a^2)(b^2-x^2)}}dx = z.\,\Gamma\,(½ - i\omega)\Gamma\left(\frac{3}{2} + i\omega\right)F_3(½ + i\omega, ½ - i\omega, ½ - i\omega, \frac{3}{2} + i\omega, 2, z, -z)$

$+ \frac{a}{a+b}\,\Gamma\,(½ - i\omega)\,\Gamma\,(½ - i\omega)F_3(½ + i\omega, ½ - i\omega, ½ - i\omega, 1, z, -z)$ (6.34)

$$\int_a^b \frac{x^2 e^{i\omega\theta}}{\sqrt{(x^2-a^2)(b^2-x^2)}}dx = \frac{\pi a^2}{(a+b)\cosh \pi\omega} F_3(½ + i\omega, ½ - i\omega, ½ - i\omega, ½ + i\omega, 1, z, -z)$$

$$+ (b - a)\, \Gamma\, (½ - i\omega)\, \Gamma\, (\tfrac{3}{2} + i\omega) F_3(½ + i\omega, ½ - i\omega, \tfrac{3}{2} + i\omega, 2, z, -z)$$

$$- \frac{z(b-a)}{b}\, \Gamma\left(\frac{3}{2} - i\omega\right)\, \Gamma\left(\frac{3}{2} + i\omega\right) F_3(½ + i\omega, ½ - i\omega, \tfrac{3}{2} - i\omega, 3, z, -z) \qquad (6.35)$$

where z = (b-a)/(b+a) and F_3 is hypergeometric function of two variables defined in (14, p. 1053).

Now separating real and imaginary parts we get

$$I_0 = \frac{\pi}{(a+b)\cosh \pi\omega} 2F_1(½ + i\omega, ½ - i\omega, 1, z^2) \qquad (6.36)$$

$$I_1 = \frac{\omega\pi z}{\cosh \pi\omega} F_3(½ + i\omega, ½ - i\omega, ½ - i\omega, 2, z, -z) \qquad (6.37)$$

$$I_2 = \frac{\pi a^2}{(a+b)\cosh \pi\omega} F_3(½ - i\omega, ½ + i\omega, ½ + i\omega, ½ - i\omega, 1, z, -z)$$

$$+ \frac{\pi\,(b-a)}{2\cosh \pi\omega} F_3(½ - i\omega, ½ + i\omega, ½ + i\omega, ½ - i\omega\ 2, z, -z)$$

$$- \frac{\pi(b-a)^2}{4(a+b)\cosh \pi\omega} \sum_{p=0}^{\infty} \frac{(½ - i\omega)_p (½ + i\omega)_p}{3!(3)_p}\, .\, z^p \qquad (6.38)$$

For 0 < x < a, we have

$$\sigma_{yy}(x, 0+) + i\,\sigma_{xy}(x, 0+) + m_{\phi y}(x, 0+) = p_0[1 + \{(a^2 - x^2)(b^2 - x^2)\}^{-½}(x^2 + h'_2 - ih''_1 x)x\, e^{-iw\theta}] \qquad (6.39)$$

If N_1, N_2 and N_3 be the normal and shear stress intensity factors at the crack tip x = a, then they are given by the equation

$$N_1 + iN_2 + N_3 = \lim_{X\to a} \left[\sqrt{(a-x)}\{\sigma_{yy}(x, 0+) + i\,\sigma_{xy}(x, 0+) + m_{\phi y}(x, 0+)\}e^{i\omega\theta}\right]$$

$$= - p_0[2a\,(b^2 - a^2)]^{-½}(a^2 - iah''_1 + h'_2) \qquad (6.40)$$

Similarly for x > b

$$\sigma_{yy}(x, 0+) + i\,\sigma_{xy}(x, 0+) + m_{\phi y}(x, 0+) = P_0[1 + \{(x^2 - x^2)(x^2 - b^2)\}^{-½}(h''_2 + h''_1\, x - x^2)e^{-iw\theta}] \qquad (6.41)$$

If N_1, N_2 and N_3 be the normal and shear stress intensity factors at the crack tip x = b, then they are given by the equation

$$N_1 + iN_2 + N_3 = \lim_{X\to b} ((x - b)^{½} \{\sigma_{yy}(x, 0+) + i\,\sigma_{xy}(x, 0+) + m_{\phi y}(x, 0+)\}\, e^{i\omega\theta})$$

$$= -P_0[2b\,(b^2 - a^2)]^{-½}(h'_2 + h''_2\, b - b^2) \qquad (6.42)$$

Case III. Three Collinear Griffith Cracks

Here we shall consider the case in which three collinear Griffith cracks are located at the interface. Also we take the prescribed pressure p(x) = P_0 (a constant). Let the three cracks a, b, c, are positive number such that a < b < c.

From equation 4.20) we have

$$X(z) = \{(z^2 - a^2)\,(z^2 - b^2\,(z^2 - c^2)\}^{-½} \left[\frac{(z+a)(z-b)(z+c)}{(z-a)(z+b)(z-c)}\right]^{i\omega} \qquad (7.1)$$

$$P(z) = (h_1 z^2 + h_2 z + h_3) \qquad (7.2)$$

And

$$f(z) = [(a_1c_2 - a_2c_1)(b_2c_3 - b_3c_2) - (a_2c_3 - a_3c_2)\,(b_1c_2 - b_2c_1))\, \frac{P_0}{2\mu i}$$

$= f_0$ say in (4.22), we get

$$L(z) = f_0\,(z^3 + g_1 z^2 + gz + g_3) \qquad (7.3)$$

where g_1, g_2, g_3 are known constants given below:

$$g_1 = -2\, i\omega(a + c - b)(a^2 + b^2 + c^2)$$

$$g_2 = -2\{\omega^2(b^2 - 2ab - 2bc + 2ac) + a^2 + c^2\}$$

$$g_3 = i\omega\left\{\frac{2}{3}(1 - 2\omega^2)(c^3 + a^3 - b^3) - 4(a + c)(ac + b^2\omega^2) - 4b\,(c^2 + a^2) - 8abc\,\omega^2\right\}$$

From (4.23) we obtain

$$\Lambda(z) = f_0 \frac{2}{b(\xi)}\left[1 - (z^3 + h_1 z^2 + h_2 z + h_3)\, X(z)\right] \quad (7.4)$$

Following values of $X^+(x)$ and $X^-(x)$ are obtained by (58) (a) for the relation b < x < c.

$$X^+(x) = -K\,X^-(x) = \frac{i\,K^{1/2}(\cos\omega\theta_1 + i\sin\omega\theta_1)}{\{(a^2-x^2)(x^2-b^2)(c^2-x^2)\}^{1/2}} \quad (7.5)$$

(b) For the relation $0 < x < a$

$$X^+(x) = -K\,X^-(x) = \frac{i\,K^{1/2}(\cos\omega\theta_1 + i\sin\omega\theta_1)}{\{(a^2-x^2)(b^2-x^2)(c^2-x^2)\}^{1/2}} \quad (7.6)$$

(c) For the relation $a < x < b$

$$X^+(x) = X^-(x) = -\frac{(\cos\omega\theta_2 + i\sin\omega\theta_2)}{\{(x^2-a^2)(b^2-x^2)(c^2-x^2)\}^{1/2}} \quad (7.7)$$

(d) For the relation $x > c$

$$X^+(x) = X^-(x) = \frac{(\cos\omega\theta_2 + i\sin\omega\theta_2)}{\{(x^2-a^2)(x^2-b^2)(x^2-c^2)\}^{1/2}} \quad (7.8)$$

Where

$$\theta_1 = \log\left[\frac{(x+a)(x-b)(x+c)}{(x-a)(x+b)(c-x)}\right]$$

$$\theta_2 = \log\left[\frac{(x+a)(b-x)(x+c)}{(x-a)(x+b)(x-c)}\right]$$

Now from the equations (2.8),(2.9), (2.10),(4.14) we get-

$$\lambda(x) = \Lambda^+ - \Lambda^- = i\{(a_1c_2 - a_2c_1)(b_2c_3 - b_3c_2) - (a_2c_3 - a_3c_2)(b_1c_2 - b_2c_1)\}$$
$$\{2b(\xi)\}^{-1}(x^3 + h_1x^2 + h_2x + h_3)(X^+ - X^-) \quad (7.9)$$

From (7.9) we get

$$\Lambda^+(x) - \Lambda^-(x) = -f_0\left[\left(\frac{\{b(\xi)-c(\xi)\}^2 - a(\xi)^2}{\{b(\xi)^2-c(\xi)\}^2 + a(\xi)^2}\right)\{(a^2-x^2)(b^2-x^2)(c^2-x^2)\}^{-1/2}(x^3 + h_1x^2 + h_2x + h_3)(\cos\omega\theta_1 + i\sin\omega\theta_1)\right] \quad (7.10)$$

$$\Lambda^+(x) + \Lambda^-(x) = if_0\left[1 - i\left(\frac{\{c(\xi)-b(\xi)\}^2 - a(\xi)^2}{\{c(\xi)^2-b(\xi)\}^2 + a(\xi)^2}\right)\{(a^2-x^2)(b^2-x^2)(c^2-x^2)\}^{-1/2}(x^3 + h_1x^2 + h_2x + h_3)(\cos\omega\theta_1 + i\sin\omega\theta_1)\right] \quad (7.11)$$

Now taking

$$h_J = h_J + ih_J,\ J = 1,2,3 \quad (7.12)$$

where h_j, and $h_{j''}$ are real constants using equations (4.12),(4.18), (7.5), (7.6), (7.9), (7.12) we get

(a) for b < x < c

$$s(x) = -f_0\left\{\left(\frac{c(\xi) - b(\xi) + a(\xi)}{c(\xi)\ldots - b(\xi) - a(\xi)}\right)\right\}^{1/2}\{(a^2-x^2)(b^2-x^2)(c^2-x^2)\}^{-1/2}$$
$$\{(x^3 + h'_1x^2 + h'_2\,x + h'_3)\cos\omega\theta_1 - (h''_1x^2 + h''_2\,x + h''_3)\sin\omega\theta_1\}$$

$$r(x) = f_0\left[\left(\frac{c(\xi) - b(\xi) + a(\xi)}{c(\xi) - b(\xi) - a(\xi)}\right)\right]^{1/2}\{(a^2-x^2)(b^2-x^2)(c^2-x^2)\}^{-1/2}$$
$$\{(x^3 + h'_1x^2 + h'_2\,x + h'_3)\sin\omega\theta_1 - (h''_1x^2 + h''_2\,x + h''_3)\cos\omega\theta_1\}$$

$$w(x) = -f_0\left[\left(\frac{c(\xi) - b(\xi) + a(\xi)}{c(\xi) - b(\xi) - a(\xi)}\right)\right]^{1/2}\{(a^2-x^2)(b^2-x^2)(c^2-x^2)\}^{-1/2}$$
$$\{(x^3 + h'_1x^2 + h'_2\,x + h'_3)\cos\omega\theta_1 + (h''_1x^2 + h''_2\,x + h''_3)\sin\omega\theta_1\} \quad (7.13)$$

(b) for 0 < x < a

$$s(x) = -f_0\left[\left(\frac{c(\xi) - b(\xi) + a(\xi)}{c(\xi) - b(\xi) - a(\xi)}\right)\right]^{+1/2}\{(a^2-x^2)(b^2-x^2)(c^2-x^2)\}^{-1/2}$$

$$r(x) = f_0\left[\left(\frac{c(\xi) - b(\xi) + a(\xi)}{c(\xi) - b(\xi) - a(\xi)}\right)\right]^{+1/2}\{(a^2-x^2)(b^2-x^2)(c^2-x^2)\}^{-1/2}$$
$$\{(x^3 + h'_1x^2 + h'_2\,x + h'_3)\sin\omega\theta_2 + (h''_1x^2 + h''_2\,x + h''_3)\cos\omega\theta_2\}$$

(7.14)

$w(x) = f_0\left[\left(\frac{c(\xi)-b(\xi)+a(\xi)}{c(\xi)-b(\xi)-a(\xi)}\right)\right]^{½}\{(a^2-x^2)(b^2-x^2)(c^2-x^2)\}^{-½}$

$\{(x^3+h'_1x^2+h'_2\,x+h'_3)\cos\omega\theta_2 + (h''_1x^2+h''_2\,x+h''_3)\sin\omega\theta_2\}$

Now since r(x), s(x) and w(x) are odd and even functions of x respectively, we must have

$h'_1 = h'_2 = h'_3 = 0$ (7.15)

from the equations (4.14), (4.12), (2.8 – 2.10), we get for $a \le x \le b$, $x > c$.

$\Lambda^-(x) = \Lambda^+(x) = \frac{1}{2\pi i}\int_L \frac{\lambda(u)}{(u-x)}\,du$

Or

$\Lambda^-(x) = \Lambda^+(x) = \frac{1}{2\pi i}\int_L \frac{s(u)-i\,r(u)+w(u)}{(u-x)}\,du = i\,f_0\frac{2}{b(\xi)}[1+(x^3h_1x^2+hx_2+h_3)X^+(x)]$ (7.16)

Substituting the values of $X^+(x)$ from (7.7) and(7.8) and using (4.4) we have

for $a < x < b$

$\sigma_{yy}(x, 0+) = \sigma_{xy}(x, 0-) = -P_0\,(1+\{(x^2-a^2)(b^2-x^2)(c^2-x^2)\}^{-½}\{(x^3h'_2\,x)\cos\omega\theta - (h''_1x^2+h''_3)\sin\omega\theta\})$ (7.17)

$\sigma_{xy}(x, 0+) = \sigma_{xy}(x, 0-) = P_0[\{(x^2-a^2)(b^2-x^2)(c^2-x^2)\}^{-½}\{(x^3h'_2\,x)\sin\omega\theta + (h''_1x^2+h''_3)\cos\omega\theta\}]$ (7.18)

$m_{\phi y}(x, 0+) = m_{\phi y}(x, 0-) = -P_0[1+\{(x^2-a^2)(b^2-x^2)(c^2-x^2)\}^{-½}\{(x^3h'_2\,x)\cos\omega\theta - (h''_1x^2+h''_3)\sin\omega\theta\}]$ (7.19)

for $x > c$

$\sigma_{yy}(x, 0+) = \sigma_{xy}(x, 0-) = P_0[1-\{(x^2-a^2)(x^2-b^2)(x^2-c^2)\}^{-½}\{(x^3h'_2\,x)\cos\omega\theta - (h''_1x^2+h''_3)\sin\omega\theta\}]$ (7.20)

$\sigma_{xy}(x, 0+) = \sigma_{xy}(x, 0-) = -P_0[\{(x^2-a^2)(x^2-b^2)(x^2-c^2)\}^{-½}\{(x^3h'_2\,x)\sin\omega\theta - (h''_1x^2+h''_3)\cos\omega\theta\}]$ (7.21)

$m_{\phi y}(x, 0+) = m_{\phi y}(x, 0-) = -P_0[1+\{(x^2-a^2)(x^2-b^2)(x^2-c^2)\}^{-½}\{(x^3h'_2\,x)\cos\omega\theta + (h''_1x^2+h''_3)\sin\omega\theta\}]$ (7.22)

for large values of x, using 6.17 – 6.6.22) we get

$\sigma_{yy}(x, 0+) = 0\,(x^2)$

$\sigma_{xy}(x, 0+) = P_0\,\{2\omega(a+c-b)+h''_1\}x^{-1}+0(x^{-2})$

$m_{\phi y}(x, 0+) = 0\,(x^{-2})$

Assuming stress components to be $0(x^{-1})$ for large x we get

$h''_1 = -2\omega\,(a+c-b)$ (7.23)

Other constants h'_2 and h''_3 can be calculated by using the condition (4.26), from it we obtain

$$\int_b^c \frac{(x^2+ih''_1x^2+h''_2+ih''_3)(\cos\omega\theta - i\sin\omega\theta)}{(x^2-a^2)(x^2-b^2)(c^2-x^2)}dx = 0$$

(7.24)

$N_a^1+N_a^2+N_a^3 = \lim_{X\to\infty}((a-x)^{½}\{\sigma_{yy}(x,0+) + i\,\sigma_{xy}(x,0+) + m_{\phi y}(x,0+)\}e^{i\omega\theta})$

for $x > c$

$N_a^1+N_a^2+N_a^3 = \lim_{X\to\infty}((x-c)^{½}\{\sigma_{yy}(x,0+) + i\,\sigma_{xy}(x,0+) + m_{\phi y}(x,0+)\}e^{i\omega\theta})$

Similarly the normal and shear stress intensity factors can be calculated at the crack tip x = b and x = c.

RESULTS AND DISCUSSION

In this paper, thermal stress field due to a system of Griffith cracks lying at the interface of two bonded dissimilar micropolar elastic half planes is considered. A general formulation of a Griffith cracks are situated with respect to y-axis and taken perpendicular to the interface has been considered. The problem is first reduced to a system of simultaneous dual integral equations which are further reduced to the solution of Riemann boundary value problem. Expressions for evaluating the stress intensity factors at the crack tip are derived. Calculations have been done for collinear cracks when constant temperature is prescribed on the crack surfaces. (Bregman and Kassir, 1974; Ejike and Sneddon, 1969; Gerasoulis and Srivastav, 1980; Gradshteyn and Ryzhik, 1965; Green and Zerna, 1960; Lowengrub and Srivastava, 1968; Sneddon and Lowengrub, 1969; Srivastava *et al.*, 1977; Srivastava *et al.*, 2000; Srivastava and Lowengrub, 1970; Tranter, 1961; Yadava *et al.*, 2002).

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