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FIXED POINT THEOREM IN BANACH SPACE

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ABSTRACT

In this paper, we established fixed point theorem with help of self mapping which satisfying contractive type of condition in Banach space.

KEYWORDS: Fixed point, Banach space, contraction mapping

In Bajaj (2001) suggested some work for the contractive type of mapping for the fixed point theorems and also give some idea by Ciric (1977), Fisher (1979) and Chaubey & Sahu (2011) for find fixed point.

By help of Fisher (1979) take a mapping $f:X\rightarrow X$ satisfying the condition of type

$$[d(fx, fy)]^2 \le \alpha d(x, fx)d(y, fy) + \beta d(x, fy)d(y, fx) \quad (1)$$

for all $x, y \in X$ and $0 \le \alpha < 1$ and $\beta \ge 0$

Theorem: Let X be a closed and convex subset of a Banach Space and let f be a self mapping of X into itself which satisfies the following condition:

$$[d(fx,fy)]^2 \le \alpha \,.\, \min\begin{pmatrix} \frac{1}{5}\{d(x,fx)d(x,fy) + d(x,fy)d(y,fx)\}, \\ \frac{1}{5}\{d(x,fx)d(x,fy) + d(x,fx)d(y, fx)\}, \\ \frac{1}{5}\{d(x, fy)d(y, fx) + d(x, fx)d(y, fx)\} \end{pmatrix} \quad (2)$$

for all $x \in X$ and $y \in \{fx, gx, fgx\}$ and $0 \le \alpha < 1$

where g is self mapping in X such that

$$gx = \frac{x + fx}{2} \quad (3)$$

Then f has a fixed point

Proof: By the definition of metric space

$$d(x,fx) = ||x-fx|| = 2||x - \left(\frac{fx + x}{2}\right)||$$

$$= 2||x - gx||$$

$$d(x,fx) = 2d(x, gx) \quad (4)$$

$$d(fx,gx) = ||fx-gx|| = ||fx-\left(\frac{x + fx}{2}\right)||$$

$$= ½ ||fx - x||$$

$$d(fx,gx) = ½d(x, fx) \quad (5)$$

$$= d(x,gx)$$

Taking

$$p = 2(gx - fgx) + fgx \qquad [fg \approx fog]$$

$$p = 2\left(\frac{x + fx}{2} - fgx\right) + fgx$$

$$p = x + fx-2fgx + fgx$$

$$p = x + fx - fgx \quad (6)$$

Now

$$d(p, fgx) = ||p-fgx|| = ||x-fgx + fx - fgx||$$

$$= ||x+fx-2fgx|| = ||2gx -2fgx||$$

$$= 2d(gx, fgx) = 2.2d(gx, ggx) \qquad \text{by (4)}$$

$$d(p, fgx) = 4d(gx, g^2x) \quad (7)$$

Since $d(p, fgx) \le d(p, fx) + d(fx, fgx)$

$$= ||x-fgx + fx - fx|| + d(fx, fgx)$$

$$\le d(x, fx) + 2d(fx, fgx) \quad (8)$$

from (7) and (8), we have

$$4d(gx, g^2x) \le d(x, fx) + 2d(fx, fgx)$$

$$4d(gx, g^2x) \le 2d(x, gx) + 2d(fx, fgx)$$

$$2d(gx, g^2x) \le d(x, gx) + d(fx, fgx) \quad (9)$$

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from (2)

$$[d(fx,fgx)]^2 \le \alpha \,.\, \min\begin{pmatrix} \frac{1}{5}\{d(x,fx)d(x,fgx) + d(x,fgx)d(gx,fx)\}, \\ \frac{1}{5}\{d(x,fx)d(x,fgx) + d(x,fx)d(gx, fx)\}, \\ \frac{1}{5}\{d(x, fgx)d(gx, fx) + d(x, fx)d(gx, fx)\} \end{pmatrix}$$

Using triangle inequality,

$$[d(fx,fg\,x)]^2 \le \alpha \,.\, \min\,[\tfrac{1}{5}\{d(x,fx)\{d(x,fx) + d(fx,fgx)\} + d(gx,fx)\{d(x,fx) + d(fx,fgx)\}\},$$

$$\tfrac{1}{5}\{d(x, fx)\{d(x, fx) + d(fx, fgx)\} + d(x, fx)\, d(fx,gx)\},$$

$$\tfrac{1}{5}\{d(fx, gx)\{d(x, fx) + d(fx, fgx)\} + d(x, fx)\, d(fx,gx)\}]$$

$$[d(fx,fgx)]^2 \le \alpha \,.\, \tfrac{1}{5}\ [d(fx,gx)\{d(x,fx) + d(fx,fgx)\} + d(x,fx)d(fx,gx)]$$

$$= \alpha/5\ d(fx,gx)[d(x,fx) + d(fx,fgx) + d(x,fx)]$$

$$= \alpha/5\ d(fx,gx)[2d(x,fx) + d(fx,fgx)] \qquad \text{By (5)}$$

$$= \alpha/5\ d(fx,gx)[4d(x,gx) + d(fx,fgx)]$$

$$5[d(fx,fgx)]^2 \le 4\alpha\ [d(x,gx)]^2 + \alpha d(x,gx)\ d(fx,fgx)]$$

$$5[d(fx,fgx)]^2 - \alpha\ d(x,gx)\ d(fx,fgx) - 4\alpha[d(x,gx)]^2 \le 0 \qquad (10)$$

Now $ax^2 + bx + c \le 0$

then $x - \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) \le 0$ (take + ve sign)

so $d(fx,fgx) - \xi d(x, gx) \le 0$

where $\xi = \frac{\alpha + \sqrt{\alpha^2 + 80\alpha}}{10} < 1$, since $0 \le \alpha < 1$.

$$d(fx,fgx) \le \xi d(x, gx) \qquad (11)$$

where $0 \le \xi < 1$

using (9) and (11), we have

$$2d(gx, g2x) \le d(x, gx) + d(fx, fgx)$$

$$2d(gx, g2x) \le d(x, gx) + \xi d(x, gx)$$

$$= (1+\xi)\, d(x, gx)$$

$$d(gx, g^2x) \le \frac{(1+\xi)}{2}\ d(x, gx) \qquad (12)$$

Similarly

$$d(g^2x, g^3x) \le \left(\frac{1+\xi}{2}\right) d(gx, g^2x)$$

$$\le \left(\frac{1+\xi}{2}\right)^2 d(x, gx)$$

....

$$d(g^nx, g^{n+1}x) \le \left(\frac{1+\xi}{2}\right) d(g^{n-1}x, g^nx)$$

$$\le \ldots\ldots\ldots\ldots \le \left(\frac{1+\xi}{2}\right)^n d(x, gx)$$

$$d(g^nx, g^{n+1}x) \le \left(\frac{1+\xi}{2}\right)^n d(x, gx) \qquad (13)$$

Since $\xi < 1 \Rightarrow \frac{1+\xi}{2} < 1$ then in (13), R.H.S. tends to zero as $n \to \infty$. Then by definition of Cauchy sequence $\{g^n x\}_{n=0}^{\infty}$ is a Cauchy sequence. Since X is a Banach space so by property of completeness $\{g^n x\}_{n=0}^{\infty}$ is convergent to a fixed point. Then there exist some element ν in X such that

$\lim_{n\to\infty} g^n x = \nu$ and the sequence $\{g^n x\}_{n=0}^{\infty}$ converges to ν

Now consider,

$$d(\nu, f\nu) \le d(\nu, g^{n+1}x) + d(g^{n+1}x, f\nu)$$

$$= d(\nu, g^{n+1}x) + d(gg^nx, f\nu)$$

$$= d(\nu, g^{n+1}x) + \|gg^nx, f\nu\|$$

$$= d(\nu, g^{n+1}x) + \|\tfrac{1}{2}(fg^nx + g^nx) - f\nu\|$$

$$\le d(\nu, g^{n+1}x) + \tfrac{1}{2}d(g^nx, f\nu) + \tfrac{1}{2}d(fg^nx, f\nu)$$

$$= d(\nu, g^{n+1}x) + \tfrac{1}{2}d(g^nx, fg^nx)$$

$$= d(\nu, g^{n+1}x) + d(g^nx, g^{n+1}x)$$

So, $d(\nu, f) \le d(\nu, g^nx)$ as $n \to \infty$

$d(\nu, f) \le 0$

so $d(\nu, f\nu) = 0$

so $f\nu = \nu$

Hence f has a fixed point

Hence proved the theorem

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