SOME DIFFERENT STROKES IN CAUCHY'S THEOREM

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ABSTRACT

In this paper I tried to give some different strokes on theorem given by Cauchy's on convergence and first limit theorem. We have some proofs on the same but I found some of mathematics learner found difficult to understand the theorem. Merely four months on working on it I came to this conclusion that we may use the proposed approach too. Therefore this proof will help such mathematics learner a lot.

KEYWORDS: Limit, Cconvergence, Cauchy's criteria, Cauchy's theorem.

In many centuries, study of sequence and series is major part in real analysis and specially in seventeenth century Euler's, Cauchy's are notable mathematician those who have given many theorems, proposition & criteria in the field of real analysis.

Many researchers have been done their research work on these mathematicians' works. Here I tried to give one more strokes in proving the two theorems.

Cauchy has given first theorem on limit but the proof which I delivered to my students I found few are asking why to choose $a_n = b_n + l$ do we have any other method that's why I removed this part and provide another way to proof. Similarly with different approach the theorem "every convergent sequence is Cauchy's sequence" is also proved.

Theorem 1 and theorem 2 are existing mode of proof whereas theorem 3 and theorem 4 are proposed method of proof which I used to call proof with different strokes.

Definition 1

The sequence $\{a_n\}$ converges (has limit l) to l when this holds: for any ε > 0 there exists K such that $|a_n - l| < \varepsilon \;\; \forall \; n \geq K$.

Informally, this says that as n gets larger and larger the numbers $\{a_n\}$ get closer and closer to l.

Definition 2

A sequence $\{a_n\}$ is bounded above if there is a real number b such that

$$\{a_n\} \leq b \; for \; all \; n$$

and bounded below if there is a real number c such that

$$\{a_n\} \geq c \; for \; all \; n$$

or bounded if there is a real number r such that

$$|a_n| \leq r \; for \; all \; n$$

Definition 3

We say that a sequence of real numbers $\{a_n\}$ is a Cauchy sequence provided that for every ε > 0, there is a natural number N so that when n;m ≥ N, we have that:

$$|a_n - a_m| < \varepsilon$$

Cauchy Criterion (or Cauchy Theorem)

Suppose a sequence $\{a_n\}$ converges. Then for any ε > 0, there is N, such that m; n > N =)

$$|a_n - a_m| < \varepsilon.$$

Proof:

Suppose $\lim_{n\to\infty} a_n = l$ For any ε > 0, there is N, such that n > N implies

$|a_n - l| \leq \frac{\varepsilon}{2}$. Then m; n > N implies

$$|a_n - a_m| = |a_n - l + l - a_m| \leq |a_n - l| + |a_m - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Theorem 1: Every convergent sequence is Cauchy sequence

Proof:

Suppose $\{a_n\}$ is a convergent sequence, and Let for all n≥m, $\lim_{n\to\infty} a_n = l$.

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We can find n of N such that for all n> N,

$|a_n - l| \le \frac{\varepsilon}{2}$. Therefore, by the triangle inequality, for all m, n> N,

$|a_m - a_n| \le |a_m - l| + |l - a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

So $\{a_n\}$ is Cauchy.

Theorem 2: Cauchy's first theorem on limit

If $\{a_n\}$ is a sequence of real number and $\lim_{n\to\infty} a_n = l$ then

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \ldots\ldots + a_{n-1} + a_n}{n} = l$$

Proof:

Let $a_n = b_n + l \ \ \forall n \in N$

Since $\lim_{n\to\infty} a_n = l$

Therefore

$$l = \lim_{n\to\infty} a_n = \lim_{n\to\infty}(b_n + l) = \lim_{n\to\infty} b_n + l$$

So $\lim_{n\to\infty} b_n = 0$ as well as

$$\frac{a_1 + a_2 + \ldots\ldots a_n}{n} = l + \frac{b_1 + b_2 + \ldots\ldots b_n}{n} \quad \ldots..(1)$$

Since $\lim_{n\to\infty} b_n = 0$ therefore for $\varepsilon > 0 \ \exists m \in N$ s.t.

$$\forall n \ge m \ \Rightarrow |b_n - 0| < \frac{\varepsilon}{2} \quad \ldots..(2)$$

Again since $\lim_{n\to\infty} b_n = 0$ therefore <b_n> is convergent sequence and hence it is bounded

So $\exists K > 0 \ \ s.t. \ |b_n| \le K \ \ \forall n \in N$ …..(3)

Form equation (1)

$$\left|\frac{a_1 + a_2 + \ldots\ldots a_n}{n} - l\right| = \left|\frac{b_1 + b_2 + \ldots\ldots b_n}{n}\right|$$

$$\le \frac{|b_1| + |b_2| + \ldots\ldots |b_n|}{n}$$

$$= \frac{|b_1| + |b_2| + \ldots\ldots |b_m|}{n} + \frac{|b_{m+1}| + |b_{m+2}| + \ldots\ldots |b_n|}{n}$$

$$< \frac{m}{n}K + \frac{n-m}{n}.\frac{\varepsilon}{2} \quad \forall n \ge m$$

$$< \frac{m}{n}K + \frac{\varepsilon}{2}$$

If $\frac{m}{n}K < \frac{\varepsilon}{2} \ \Rightarrow n > \frac{2m}{\varepsilon}K$ then the above inequality become

$$\left|\frac{a_1 + a_2 + \ldots\ldots a_n}{n} - l\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

For all

$$n > \max(m, \frac{2m}{\varepsilon}K)$$

$$\therefore \ \lim_{n\to\infty} \frac{a_1 + a_2 + \ldots\ldots a_n}{n} = l$$

Theorem 3: Every convergent sequence is Cauchy sequence.

Proof:

Suppose we have a sequence $\{a_n\}$ converging to l, that means $\lim_{n\to\infty} a_n = l$ or we may say

$$|a_n - l| < \frac{\varepsilon}{2} \quad \forall \ \ n \ge m \quad \ldots..(1)$$

Therefore

$$|a_{n+1} - l| < \frac{\varepsilon}{2}$$

$$|a_{n+2} - l| < \frac{\varepsilon}{2}$$

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$$|a_{n+p} - l| < \frac{\varepsilon}{2}$$

$$\forall \ \ n \ge m \quad and \quad p \ge 0$$

Now for Cauchy sequence

$$\begin{aligned}|a_{n+p} - a_n| &= |a_{n+p} - l + l - a_n| \\ &\leq |a_{n+p} - l| + |l - a_n| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

Therefore

$$|a_{n+p} - a_n| < \varepsilon \quad \forall \; n \geq m \quad and \quad p \geq 0$$

Hence $\{a_n\}$ in Cauchy's sequence.

Theorem 4: Cauchy's first theorem on limit

If $\{a_n\}$ is a sequence of real number and

$\lim_{n\to\infty} a_n = l$ **then**

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \ldots\ldots + a_{n-1} + a_n}{n} = l$$

Proof:

Suppose we have a sequence $\{a_n\}$ having limit l , that means $\lim_{n\to\infty} a_n = l$ or we may say

$$|a_n - l| < \varepsilon \qquad \forall \; n \geq m \qquad \ldots..(1)$$

Equation (1) can be interpreted as

$$l - \varepsilon < a_n < l + \varepsilon$$

In particular ε =1

$$|a_n - l| < 1 \qquad \forall \; n \geq m$$

So

$$|a_n| = |a_n - l + l| \leq |a_n - l| + |l| < 1 + |l|$$

Let $M = \max\{|a_1|, |a_2|, |a_3| \ldots\ldots.. |a_{m-1}|, 1 + |l|\}$

$$|a_n| \leq M \qquad \forall \; n \in N$$

Therefore a_n is bounded

Since

$$l - \varepsilon < a_n < l + \varepsilon$$

For n= 1,2,3,4………m,m+1, m+2, m+3,………..n the above inequality is as follows

$$-M < a_1 < M$$
$$-M < a_2 < M$$
$$-M < a_3 < M$$

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$$-M < a_m < M$$
$$l - \varepsilon < a_{m+1} < l + \varepsilon$$
$$l - \varepsilon < a_{m+2} < l + \varepsilon$$

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$$l - \varepsilon < a_n < l + \varepsilon$$

Adding all the above term we have

$$-mM + (n-m)(l-\varepsilon) < a_1 + a_2 + \ldots + a_n < mM + (n-m)(l+\varepsilon)$$

Dividing above inequality by n

$$\frac{-mM + (n-m)(l-\varepsilon)}{n} < \frac{a_1 + a_2 + \ldots + a_n}{n} < \frac{mM + (n-m)(l+\varepsilon)}{n}$$

$$\frac{m}{n}(-M) + \frac{(n-m)(l-\varepsilon)}{n} < \frac{a_1 + a_2 + \ldots + a_n}{n} < \frac{m}{n}M + \frac{(n-m)(l+\varepsilon)}{n}$$

Now taking $n \to \infty$ the above inequality become

$$l - \varepsilon < \frac{a_1 + a_2 + \ldots\ldots + a_{n-1} + a_n}{n} < l + \varepsilon$$

Or we can say

$$\left| \frac{a_1 + a_2 + \ldots\ldots + a_{n-1} + a_n}{n} - l \right| < \varepsilon \qquad \forall \; n \geq m$$

i.e. $\lim_{n\to\infty} \frac{a_1 + a_2 + \ldots\ldots + a_{n-1} + a_n}{n} = l$

CONCLUSION

On the basis of proof of above four theorems we may come to this conclusion that the proof of theorem 1 & 2 also may have different solution or proof in theorem 1 existing proof on the basis of concept of convergence whereas in theorem 3 taking the concept of limit. In continuation theorem 2 sequence is represented by other sequence and the proof done but in theorem 4 limit & bounded concept was taken. The proposed methods are also one of the easy & different way to conclude theorems.

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